Report on

Term Structure Models of Commodity Prices:

Elaboration and Improvement

(Confidential)

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**Introduction**

The aim of this study on term structure models of commodity prices is to develop tools for the management of the risk associated with most of energy markets.

Understanding the mechanisms of the formation of commodity prices is a prerequisite for the use of derivatives instruments to hedge the risks. The analysis of these mechanisms, in D. Lautier’s Ph.D. thesis, has lead to the elaboration of a new term structure model.

The term structure of futures prices describes the relationships between the spot price and futures prices for different delivery dates. It is supposed to resume all the information needed to hedge positions on the physical market or to make arbitrage operations. The objective of a term structure model is twofold. Firstly, it is to reproduce, as accurately as possible, the prices curve empirically observed. Secondly, it aims to prolong the curve for very long maturities, even for delivery dates which are not available in the market.

This new model, developed in the Ph.D thesis, is called the asymmetrical model. It takes into consideration the specific imperfection of the raw material markets. This imperfection is represented by the difficulty to undertake, in some circumstances, arbitrage operations between the physical and the futures markets. This difficulty is not so obvious in other financial markets, because the assets are perfectly homogeneous, there is no quality differential, and the materiality of the transactions is very low. Moreover, in the raw material markets, the possibility to make arbitrage operations are not the same when stocks are abundant than when they are rare. This has some implications on the behavior of the convenience yield. This concept was proposed for the first time in the beginning of the 40s, and it is central for the analysis of raw material term structures of futures prices. It could be defined as the comfort or the implicit gain associated with the holding of inventories. In the context of financial markets, it corresponds to the revenues given by bonds or financial stocks. The literature always supposed that the convenience yield had the same behavior when the stocks are rare and when they are abundant. The central hypothesis of the new model is to suppose that, on the contrary, it is asymmetric. It is high and strongly volatile when the physical stocks are rare, in backwardation, and it is low and quite stable in contango, when the physical stocks are abundant. The introduction of this hypothesis improves the model performances: the new model is more precise than the one proposed by Schwartz in 1997, which constitutes a reference.

The study presented in this report is a prolongation of the Ph D. thesis and of two articles (Lautier et Galli, 2000 and Lautier, 2002), which were published and are added to the report. This new research has been conducted into three main directions. First of all, its objective is to improve the understanding of the futures prices behavior, and the dynamics of the term structure of commodity prices. Secondly, the aim is to render the use of asymmetrical model more simple than it is now and to gain in calculus time. At the end of the Ph.D. thesis, the time necessary to calculate a futures price was quite important. Thirdly, we want to have a better use of the estimation methods that we employ to apply the model, namely the Kalman filters. The Kalman filters are powerful tools, which can be employed for model’s estimation in many areas in finance. They are especially well suited for finance because they are fast even if they have to deal with a large amount of information and because they allow for unobservable variables. Moreover, they can be used for linear as well as non-linear models, even if there is no analytical solution for the models.

This report is therefore divided into three parts. In the first one, we present an analysis of the term structure of commodity prices. In the second part, we present some results obtained with the asymmetrical model, namely the way the model was transformed in order to gain in calculus time. In a third part, we expose the work that was done on the estimation methods.
In the first part of the report, we analyse the crude oil prices curve, its determinants, and its dynamic. It is divided into three sections. In a first section, the analysis is based on a preliminary analysis of the data, relying on elementary statistics. Two films, representing the dynamics of the futures prices and the interest rates curves, were also made. In a second section, we propose a principal component analysis of the crude oil curve, which gives some interesting results for dynamic hedging. In the third section, we try to estimate the informational value of the futures prices and to answer the following question: how much information and which information do we need to reconstitute the term structure prices of crude oil?

The first section is centered on the underlying factors of the futures prices curves. We examined first of all the shape of the curves. Most of the time, they are in backwardation, but they are dramatically changing with time, and there is a decreasing volatilities structure. We then investigated the relative importance of three variables: the nearby futures price, the interest rates, and the physical stocks. We showed that these factors have a changing impact for different maturities. For the shorter maturities, the nearby price is the most important explaining factor. Then come the physical stocks, and the short term interest rates. For longer maturities, the nearby futures price, the physical stocks and the short term interest rates do not really matter. What is really important is the long term interest rate. Last but not least, the calculus of variograms showed us that the term structures of futures prices are almost mathematically designed.

The second section relies on principal component analysis to identify the prices curve movements and the contribution of components to volatility. This section is important because if we understand which factors explain the prices curve volatility, then we will be able to implement prices curve hedging and investment strategies. It has been shown that two components explain most of the variation in the curve. These components are the parallel shifts and the relative shifts (slope). Lastly, for long maturities only, we can identify a third component: the curvature. The same kind of analysis for the term structure of interest rates (Hull, 2000), for maturities ranging from 3 months to 30 years, leads to the conclusion that there are three factors explaining the interest rates curves.

The third section is centered on the informational value of futures prices. It confirms and gives some precision on the conclusions we reach in the first section. We show that the futures prices curve can be divided into two separate parts: the first one corresponds to maturities below 28 months, and the second part consists of maturities above 28 months. These two parts differ because the factors influencing them are of different nature and origin. The shorter part is linked to the physical market and to short term operations. For this part of the curve, the demand and supply, the inventories, and the operators expectations (for example, the fear of disruptions) should be the determinants factors. The longer part is more closely connected to investment and project financing. For this extremity of the curve, factors such as long term interest rates, but also anticipated inflation or concurrent energy prices are probably determinant. The crude oil price is presumably not mean reverting for these maturities and the convenience yield, which is so important for short maturities, should also less intervene for the longer part of the curve. Lastly, we also show that when the proper set of parameters is chosen, it is possible to reproduce the futures prices for maturities as far as seven years.

The second part of the report is dedicated to the asymmetrical model. It presents first of all the model, then the initial version of its solution, and finally it exposes the way we simplified this solution in order to reduce the calculus time. This simplification relies on an approximation. The simulations we did show that this approximation should be acceptable.

This second part is mostly centered on the calculus that were made. We added an appendix to this part, devoted to the details of the calculus. The appendix also presents simulations that were made to test the impact of the simplification.

The next step of this work is the study of calculus time. We also must verify that with this transformation, the model’s performances are still better than the ones of the Schwartz model.

The third part of the report presents the study and the comparison of three Kalman filters. These methods are useful for the estimation of the term structure models. The first filter is called the simple filter. It does only accept linear models, whereas the second and the third filters allow for non-linear models like the asymmetrical model. The second filter constitutes an extension of the simple one. Its main drawback is that it relies on an linearization of the model’s solution. Because of this
approximation, it is less precise than the previous one. We first compare these two filters and study their sensitivity to experimental conditions. Then we employ a third one: the Kushner's version of the Kalman filter. In this method, there is no explicit linearization. It is supposed that the conditional distributions, knowing the past and the measurements, are gaussian. This hypothesis is used to compute their mean and variances. Lastly, we study the performances of the two non linear filters, on a simple example, when the non linearity increases.

The main conclusions of this third part are the following. First, the extended Kalman filter introduces an approximation, which is due to the model’s linearization. This approximation has clearly an influence on the model’s performances: the extended filter leads generally to less precise estimations than the simple one. Nevertheless, until the model becomes really non linear, the difference between the two filters is quite low and the extended filter is still acceptable. The second conclusion is that the estimations results are sensible to the system’s matrix containing the errors of the measurement equation and that this matrix can be used to obtain more precise results on the estimation base. The third important conclusion is that, for the term structure models of commodity prices, the parameters are not constant in time and there is a need to recalculate them very often. This can become a problem if the model has no analytical solution, because of the computing time. The last conclusion is that a serious bias appears, with the extended filter, when the non linearity increases. Although it is not perfect, the Kushner's method is more reliable.
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Part I. Analysis of the prices curves

In this first part of the report, we analyse the crude oil prices curves, its determinants, and its dynamic. It is divided into three sections. In a first section, the analysis is first of all based on a preliminary analysis of the data, relying on elementary statistics. Two films, representing the dynamics of the futures prices and the interest rates curves, were also made. In a second section, we propose a principal component analysis of the crude oil curve, which gives some interesting results for dynamic hedging. In the third section, we try to estimate the informational value of the futures prices and to answer the following question: how much information and which information do we need to reconstitute the term structure prices of crude oil?

Section 1. Preliminary analysis of the data

In this first section, we present a preliminary analysis of the data. Our aim is two folds: first, to bring a better understanding of the behavior of commodity prices, based on simple statistics and on variograms and cross-variograms. Second, to have a first glance on the relationships between crude oil prices, interest rates, and physical stocks.

1.1. Description of the data

The database is a powerful tool for the study because it is very complete. It is divided into three parts: crude oil futures prices, American stocks of crude oil (API data), and US interest rates. They represent a large volume of information, especially because we have the curves of crude oil prices and interest rates: it represents around 100,000 futures prices, and 120,000 interest rates. The crude oil are daily settlement prices for the West Texas Intermediate futures contract of the New York Mercantile Exchange, from January 1989 to January 2002. They have been arranged such as the first futures price’s maturity corresponds actually to the one month’s maturity, such as the second futures price’s corresponds to the two months maturity, and so forth... The daily interest rates are issued from US government bonds, and the data are available for every month, from the first to the 96th, between January 1997 and February 2002. For the stocks, the weekly data represent American industrial stocks, between 1996 and 2001.

The construction of the database was the first important part of the work. The first step was to identify the data needed for the study: temporal series for crude oil prices (spot prices and futures prices for all the maturities), temporal series for interest rates (for all the available maturities), and stocks. TotalFinaElf gave most of them. The interest rates were extracted from Datastream. The second step was to check for errors in the data, and to incorporate different databases into large ones.

The description of the futures prices was the second part of the work. It has to be done because in the database, the futures do not always have the same maturity. This database covers a long period, from 1989 to 2002, during which some new futures contracts, for longer maturities, were introduced in the New York Mercantile Exchange. From the first to the 37th months, each month corresponds to the maturity of a futures contract. Then, on a longer horizon, the maturities are for the 48th, the 60th, 72nd and the 84th months. Information relative to the longer maturities is not available for the whole period and the database is divided into three parts. We have:

- all the maturities, from the first to the 30th month, between January 1989 and January 2002
- the maturities from the 31st to the 37th month between January 1989 and June 1999
- the maturities for the 48th, the 60th, the 72nd and the 84th months between June 1999 and January 2002.

1 We wish to thank TotalFinaElf who provided us with the empirical data used in this study.
1.2. Analysis of the crude oil futures prices

This preliminary data analysis is divided into three parts. Firstly, we present the behavior, on the entire period, of the crude oil prices. Secondly, we calculate and represent elementary statistics - mean, variance, correlations – for these temporal series. Thirdly, we present the variograms and the cross variograms of the futures prices.

1.2.1. The behavior of the crude oil prices on the whole period

The daily futures prices we use for this first part of the analysis covers the period between January 1989 and January 2002. Because of a lack in the data, we only used the maturities from the first to the 28th months.

The figure 1 shows a strong variability with an alternation of backwardations and contangos, and with some isolated peaks of high values in 1990, in 1997, and in 2000. It also shows that most of the time, the prices are in backwardation (except in 1998-1999), that the slope of the prices curves are sometimes (especially during the period from 2000 to 2002) important, but that sometimes also, (in 1993 and in 1995 for example) they are quite flat.

Figure 1. Futures prices for the one month to the 28th month maturities, 1989 - 2002

1.2.2. Elementary statistics

This part of the analysis is focused on the period 1997-2002 because it is common to the futures prices and the interest rates databases. We first calculated the means and the variances of the temporal series. Then we computed some correlations between futures prices.

The figure 2 confirms the fact that the crude oil market is most of the time in backwardation. The average backwardation between the 1st and the 28th maturities stands at USD 3 per barrel. It is quite large : this amount represents 15% of the average 28 months futures price.
During the period 1997-2002, the variance of the futures prices declines with the maturity. This result confirms the fact that the longer maturities are less volatile because they are less sensitive to demand and supply shocks.

As a consequence, the correlation between the first month futures prices and the other futures prices declines with the maturity, as figure 4 illustrates it. However, the relationship between the 1st month and the 28th month is still high: more than 0.82.

**Figure 2. Average futures prices for different maturities, 1997 - 2002**

**Figure 3. Variances of the futures prices for different maturities**

**Figure 4. Correlation between the first month futures prices and the futures prices for longer maturities (2 to 28 months)**
1.2.3. Variograms and cross variograms

The variogram and cross-variogram are used for a better understanding of the term structure of futures prices. The definition and the way to calculate them are presented in the appendix added to the first part of the report. To make it short, variogram give the value, for a specific maturity, of the correlation between two futures prices, as a function of the time period separating the two observations. An increase in the variogram’s value traduces a decrease in the correlation. All the variogram and cross-variogram have been computed on the period extending from June 1999 to January 2002.

The figure 5 shows the simple daily variogram for the futures prices, from 1999 to 2002.

Figure 5a. Variogram for futures prices, 1st month to 84th months maturities, 1999 - 2002

As we could have expected it, in average, the variogram for a futures price of longer maturity is greater than the variogram for a short maturity. However, what was unexpected is the fact that the picture shows an almost perfect behavior of the prices. The variogram is completely ordered with the maturities: the variogram for maturity \( \tau + 1 \) are always below the variogram for the maturity \( \tau \).

The detail for the longer maturities is shown in the figure 5b. This kind of pictures may be interpreted as the representation of two different scales of variability: a short one (of about 40 days) and a long range stationarity (more than one year), which can also be interpreted as an intrinsic structure.

Figure 5 b. Variogram for futures prices, 48,60,72,84 months maturities, between 1999 and 2002
In the figure 5c, we show the results we obtained, when the variogram are computed after the normalisation to 1 of the variances. In that case, the ordering of the variogram does not subsist.

![Variograms for futures prices with normalised variances, 1999-2002](image)

The analysis of the cross-variogram confirms the previous results we obtained with the variograms. The figure 6 represents the cross-variograms between the 1 month maturity and all the others maturities. This variogram is perfectly sorted by maturities - that is, the correlation decreases with the maturity- and depicts positive correlations for all the maturities and distance in time. The sill around 40 days and the long scale structures are still present.

![Cross-Variograms for 1 month futures price against all the others maturities, 1999-2002](image)

1.3. The relationship between crude oil prices and interest rates

Before we study of the relationship between crude oil prices and interest rates, we bring a brief review of the behavior of the interest rates.
1.3.1. The behavior of the interest rates

The interest rates curves available cover the period 1997 to 2002. There is one maturity for each month, from the first to the 95th one (some maturity were probably obtained by interpolation).

Figure 7. Interest rates for the first to the 95th delivery month, 1997 - 2002

The figure 7 shows that the behavior of the interest rates curves is different of the crude oil curves: in average, the term structure of interest rate, during 1997-2002, is in contango. The calculus of the average interest rates for different maturities, represented in figure 8, confirms this observation.

Figure 8. Average interest rates for different maturities, 1997 - 2002

The variances of the interest rates have the same behavior than the variance of the crude oil futures prices. Nevertheless, the interest rates variances seem to be less regular, less structured. And the same conclusion can be drawn for the mean.
Now, as we did with the futures prices, we will restrict the period of the study to 1999-2002. The figure 10a represents the variogram for the interest rates. The ordering is not as good it was for futures prices. First of all, the spot rates have really a different behavior from the others. Secondly, the maturities for the first to the 12th months are not strictly decreasing. They cross some of the longer maturities, like the 60 and the 72 months. Thirdly, the longer maturities are well ordered. This can be seen clearly on the figure 10b.
Interestingly, as the figure 10c illustrates it, when computed after normalising the variables to unit variances, the variogram have a completely different behavior. The maturities are sorted in increasing order, except for the maturities ranging from the 1st month to the 6th and the 12th months.

As was the case for the futures prices, the cross-variograms for interest rates (figure 11) illustrates a better organization. The correlation decreases with the maturity and the distance, except for the cross-variogram between the 2nd month and the 5th month.
Figure 11. Cross-variograms for interest rates, 2 months maturity against all the 95 other maturities, 1999-2002

Distance in time between two observations

1.3.2. The relationship between the crude oil futures prices and the interest rates

To examine the relationship between the crude oil futures prices and the interest rates, we first calculate the correlation between the two variables. In order to examine the longer maturities, the study period was reduced, from the 1st of June, 1999 to the 14th of January, 2002. The figure 12 represents three correlations. Two of them are the correlations between two interest rates of different maturity (a short term - 1 month - and a long term - 84 months) and the whole term structure of crude oil prices. The third represents the correlation between the one month futures price and the rest of the crude oil curve.

Figure 12. Correlation between interest rates and futures prices

The figure 12 leads to the following conclusions: firstly, the one month futures price has a stronger influence than the interest rates on the prices curve, and the difference is quite important. Secondly, the correlations between interest rates and futures prices are first of all positives, then they become negatives. Thirdly, the (positive) influence of the short term interest rate is stronger on the shorter futures prices maturities, and the (negative) influence of the long term interest rate is stronger on the longer futures prices maturities.
These conclusions can be completed with the table 1, representing the correlations for most of the maturities represented below:

<table>
<thead>
<tr>
<th></th>
<th>1 M.</th>
<th>3 M.</th>
<th>12 M.</th>
<th>24 M.</th>
<th>48 M.</th>
<th>60 M.</th>
<th>72 M.</th>
<th>84 M.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short term rate</td>
<td>0.61</td>
<td>0.52</td>
<td>0.26</td>
<td>0.05</td>
<td>-0.18</td>
<td>-0.24</td>
<td>-0.30</td>
<td>-0.36</td>
</tr>
<tr>
<td>Long term rate</td>
<td>0.31</td>
<td>0.25</td>
<td>-0.03</td>
<td>-0.24</td>
<td>-0.48</td>
<td>-0.53</td>
<td>-0.58</td>
<td>-0.63</td>
</tr>
<tr>
<td>Future one month</td>
<td>-0.03</td>
<td>0.98</td>
<td>0.86</td>
<td>0.70</td>
<td>0.51</td>
<td>0.44</td>
<td>0.37</td>
<td>0.29</td>
</tr>
</tbody>
</table>

This table shows that, when the futures price’s maturity reaches 60 months, the correlation with the long term rate is more important than the correlation with the future price, and this phenomenon increases with the maturity. For the longer maturity of 84 months, the influence of the long term rate is much more important than the influence of the one month future prices.

Lastly, two families of cross-variograms between interest rates and futures prices are shown. The figure 13a presents the cross-variograms between the 2 months interest rate and the futures prices ranging from the 1st to the 28th months and for the 48th, 60th, 72nd, and 84th months.

The figure 13b represents the 95 months interest rate against the same futures prices as previously. This can lead to suppose that the temporal correlations are more important that the classical correlation coefficient would show. It is also clear, on these two cross variograms, that the futures prices for maturities between 24 and 28 months break slightly the ordering, as was the case on the figures 6a and 6b, which otherwise would look more as curves obtained from a model rather than real data.

The variogram between the 2nd month interest rates and the futures prices is well ordered. For the smaller maturities, the figure 13a shows a stabilization of the correlations after approximately 45 months. For the longer maturities, the correlations progress more regularly. Most of the correlations are positive. In contrast, the variogram between the 9 month interest rate and the futures prices (figure 13b) shows that the correlations become negative when the maturities of the futures prices reaches a certain level, for all the distances. It can also be noticed that the sill around 45 months is also present for the small maturities. This suggests that, after this distance of about 45 days, a different structure of correlations in time can be observed.
Figure 13b. Cross - Variograms, 95 months Rates against 1-28, 48,60,72,84 months futures, 1999 - 2002

Distance in time between two observations

1.4. The relationship between crude oil prices and stocks

The relationship between prices and stocks has been extensively studied in the theory of commodity prices. Prices are supposed to be high when stocks are low, and conversely. The figure 14 represents the level of stocks during the period study (June 1999- December 2001) and compares it with the level of one month futures prices.

Figure 14. Stocks and futures prices, 1999 - 2001

Graphically, the inverse relationship between the level of prices and the level of stocks does not seem to be evident. It becomes evident, however, when the correlation between prices and stocks is calculated, as figure 15 shows it.
The correlation between prices and stocks is strongly negative for the shorter maturity (-0.7 for the one month delivery) and tends to zero when the maturity increases (-0.08 for the 84 months delivery).

### 1.5. Main results of the data analysis

The main conclusions we obtain in this first section of the study are the following:

- most of the time, the crude oil prices curves, between 1989 and 2002, are in backwardation
- the slope of the curves changes dramatically with time: sometimes it is high, sometimes the curves are flat
- the longer maturities of the curve are less volatile, probably because they are less sensitive to demand and supply shocks
- the term structures of futures prices are almost mathematically defined. However, considering the large number of data and the alternation of backwardations and contangs, we want to analyze separately these two kinds of term structures. Therefore, before going deeper in the analysis, in the interpretations and in the conclusions, we need to determine different time windows representative of backwardations and of contangs. It should also be interesting to include the spot price in the analysis.
- the interest rates have a positive influence on the shorter maturities and a negative influence on the end of the prices curve.
- there is a negative relationship between stocks and prices. This relationship tends to disappear for the longer maturities.

The analysis of the correlation between futures prices, interest rates and stocks leads to think that probably, different factors have an effect on the futures prices curve, depending on the maturity. The short part of the curve is strongly influenced by the prices and the stocks, whereas the long part of the curve is more influence by the long term interest rate.

### Section 2. Principal component analysis of the crude oil curve

This second section is centered on the movements of the crude oil curve. We used principal component analysis, that takes historical data on movements in the prices and attempts to define a set of components or factors that explain the movements. This kind of analysis can be very useful to determine the exposure of a crude oil company to a number of different positions in the forward market.

In this section, we will first present the basic principles of the principal component analysis. Then we will apply it to the crude oil curves and lastly we will show how it could be applied to measure the risk associated with different positions on the forward market.

#### 2.1. The principles of principal component analysis

Principal component analysis is a statistical method for reducing the dimensionality of a data set by collapsing the information. In data sets with many variables, groups of variables often move together
because they are measuring the same driving force governing the behavior of the system. In many systems, there are only a few such driving forces.

By transforming the initial matrix \((n \text{ observations} \times N \text{ variables})\) in a reduced matrix:

\[
\begin{bmatrix}
    f_{11} & f_{1M} \\
    f_{ij} & f_{jM} \\
    f_{n1} & f_{nM}
\end{bmatrix}
\]

where \(f_{ij}\) is the value of the factor \(j\) for the observation \(i\) and \(M < N\), it is possible to take advantage of this redundancy of information. The problem is simplified by replacing a group of variables with less new variables.

Principal component analysis is a quantitatively rigorous method for achieving this simplification. The method generates a new set of variables, called factors, or principal components. These factors respect two conditions:

- Linearity:
  \[ F_j = \sum_{k=1}^{N} a_{jk} X_k \]
  where \(F\) is a factor, \(X\) is the original variable, and \(a\) is a coefficient.
  In other words, each principal component is a linear combination of the original variables.

- Orthogonality:
  \[ \rho(F_j, F_m) = 0 \quad j \neq m \]
  where \(\rho\) is the correlation coefficient.
  In other words, there is no redundant information, because all the principal components are orthogonal to each other’s.

The full set of principal component is as large as the original set of variables. But it is usually the sum of the variances of the first few principal components exceed 80% of the total variance of the original data. By examining these factors, it is often possible to develop a deeper understanding of the driving forces that generated the original data.

### 2.2. Applying the principal component analysis to the crude oil prices curves

We analyzed the principal components of the crude oil prices on different periods and maturities: firstly, we analyzed the whole curve, with the longer maturities, during 1999-2002. Secondly, we analyzed the same period with shorter maturities: one month to 18 months. Thirdly, we studied the same maturities (1 to 18 months) on a longer period, from 1989 to 2002. Lastly, we made an analysis on the Gulf war period, during which the backwardation was especially strong.

The program we use to conduct the analysis is reproduced in the annex. In our study, we consider each maturity as a variable, and each date as an observation. The maturities differ because they have different volatilities. Therefore, they are standardized, in order to give each maturity the same role in the definition of the proximity between two observations.

The results we obtained on different periods are quite similar, therefore we choose to present the analysis of the whole curve on 1999-2002. We present here the components we obtained, the components scores, and the components variances.

#### 2.2.1. The components

Starting with a data set made of more than 35 variables, we choose to reduce it to 14 maturities. The full set of components is naturally as large as the original set of variables, and it is reproduced in table 2.

---

2 For the principal component analysis of term structures of interest rates, see Hull (2000).
Table 2. Components, one month’s to 84 months maturities, 1999 -2002

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Factor 1</th>
<th>Factor 2</th>
<th>Factor 3</th>
<th>Factor 4</th>
<th>Factor 5</th>
<th>Factor 6</th>
<th>Factor 7</th>
<th>Factor 8</th>
<th>Factor 9</th>
<th>Factor 10</th>
<th>Factor 11</th>
<th>Factor 12</th>
<th>Factor 13</th>
<th>Factor 14</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 month</td>
<td>0.22</td>
<td>0.46</td>
<td>0.45</td>
<td>0.51</td>
<td>0.46</td>
<td>-0.20</td>
<td>-0.08</td>
<td>0.14</td>
<td>-0.06</td>
<td>-0.03</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>3 months</td>
<td>0.25</td>
<td>0.58</td>
<td>0.33</td>
<td>-0.03</td>
<td>-0.44</td>
<td>0.39</td>
<td>-0.27</td>
<td>-0.25</td>
<td>-0.22</td>
<td>0.34</td>
<td>0.09</td>
<td>-0.15</td>
<td>0.09</td>
<td>-0.05</td>
</tr>
<tr>
<td>6 months</td>
<td>0.27</td>
<td>0.28</td>
<td>0.09</td>
<td>-0.26</td>
<td>-0.34</td>
<td>0.04</td>
<td>-0.09</td>
<td>0.11</td>
<td>0.12</td>
<td>-0.53</td>
<td>0.20</td>
<td>0.41</td>
<td>-0.31</td>
<td>0.21</td>
</tr>
<tr>
<td>9 months</td>
<td>0.29</td>
<td>0.29</td>
<td>-0.07</td>
<td>-0.31</td>
<td>-0.09</td>
<td>-0.21</td>
<td>-0.24</td>
<td>0.06</td>
<td>-0.36</td>
<td>-0.12</td>
<td>0.19</td>
<td>-0.18</td>
<td>0.52</td>
<td>-0.42</td>
</tr>
<tr>
<td>12 months</td>
<td>0.28</td>
<td>0.14</td>
<td>-0.15</td>
<td>-0.29</td>
<td>0.11</td>
<td>-0.24</td>
<td>-0.13</td>
<td>0.01</td>
<td>0.19</td>
<td>0.38</td>
<td>0.11</td>
<td>-0.37</td>
<td>-0.32</td>
<td>0.52</td>
</tr>
<tr>
<td>15 months</td>
<td>0.29</td>
<td>0.08</td>
<td>-0.22</td>
<td>-0.21</td>
<td>0.21</td>
<td>-0.16</td>
<td>0.05</td>
<td>-0.25</td>
<td>0.35</td>
<td>-0.32</td>
<td>0.30</td>
<td>-0.29</td>
<td>-0.53</td>
<td></td>
</tr>
<tr>
<td>18 months</td>
<td>0.29</td>
<td>0.03</td>
<td>-0.26</td>
<td>-0.10</td>
<td>0.22</td>
<td>-0.05</td>
<td>0.20</td>
<td>-0.08</td>
<td>-0.42</td>
<td>-0.13</td>
<td>-0.22</td>
<td>0.12</td>
<td>0.57</td>
<td>0.41</td>
</tr>
<tr>
<td>21 months</td>
<td>0.29</td>
<td>-0.02</td>
<td>-0.27</td>
<td>0.07</td>
<td>0.16</td>
<td>0.08</td>
<td>0.32</td>
<td>-0.03</td>
<td>-0.22</td>
<td>-0.42</td>
<td>0.56</td>
<td>-0.22</td>
<td>-0.29</td>
<td>-0.17</td>
</tr>
<tr>
<td>24 months</td>
<td>0.29</td>
<td>0.06</td>
<td>-0.28</td>
<td>0.03</td>
<td>0.07</td>
<td>0.01</td>
<td>-0.63</td>
<td>0.01</td>
<td>0.00</td>
<td>0.01</td>
<td>0.00</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>28 months</td>
<td>0.28</td>
<td>-0.11</td>
<td>-0.24</td>
<td>0.33</td>
<td>-0.02</td>
<td>0.14</td>
<td>0.50</td>
<td>0.01</td>
<td>0.64</td>
<td>0.11</td>
<td>-0.20</td>
<td>0.11</td>
<td>0.05</td>
<td>0.01</td>
</tr>
<tr>
<td>48 months</td>
<td>0.27</td>
<td>-0.26</td>
<td>-0.02</td>
<td>0.33</td>
<td>-0.49</td>
<td>-0.30</td>
<td>-0.04</td>
<td>0.47</td>
<td>-0.26</td>
<td>0.00</td>
<td>-0.20</td>
<td>-0.30</td>
<td>-0.01</td>
<td>-0.04</td>
</tr>
<tr>
<td>60 months</td>
<td>0.26</td>
<td>-0.32</td>
<td>0.15</td>
<td>0.16</td>
<td>-0.16</td>
<td>-0.28</td>
<td>-0.14</td>
<td>-0.26</td>
<td>-0.01</td>
<td>0.22</td>
<td>0.50</td>
<td>0.56</td>
<td>0.07</td>
<td>0.11</td>
</tr>
<tr>
<td>72 months</td>
<td>0.25</td>
<td>-0.37</td>
<td>0.29</td>
<td>-0.01</td>
<td>0.06</td>
<td>-0.11</td>
<td>-0.11</td>
<td>-0.65</td>
<td>0.04</td>
<td>-0.24</td>
<td>-0.32</td>
<td>-0.29</td>
<td>-0.10</td>
<td>-0.08</td>
</tr>
<tr>
<td>84 months</td>
<td>0.23</td>
<td>-0.42</td>
<td>0.48</td>
<td>-0.37</td>
<td>0.28</td>
<td>0.32</td>
<td>0.13</td>
<td>0.45</td>
<td>0.04</td>
<td>0.05</td>
<td>0.04</td>
<td>0.00</td>
<td>0.04</td>
<td>0.01</td>
</tr>
</tbody>
</table>

The comments will be centered on the three first factors.

The first factor, shown in the first column, corresponds to a roughly parallel shift in the prices curve: all of the values of this factor are really close to each others, and they are all positive. Therefore, whatever the maturity considered, one unit of that factor corresponds to an increase of the futures price: 25 cents for the three months futures price, 28 cents for the 12 months futures price, ... 23 cents for the 84 months futures prices. The largest weights are attributed to the 15 months to 24 months maturity, the lowest are associated with the two extremes of the curve: the one month and the 84 months maturities.

When we consider the other periods and maturities (see Appendix 2), we also observe this parallel shift of the prices curve and it is even more pronounced, because all the value of this factor are closer to each other’s. This phenomenon is especially evident for the 1989-2002 period and for all the period presented in the appendix, it can be explained by the fact that the maturities are shorter than the ones presented in table 2.

The second factor corresponds to a steepening of the prices curve. When futures prices between one to 18 months move in one direction, prices between 18 to 84 months move in the other direction. More precisely, one unit of that second factor corresponds to an increase of the shorter maturities, and to a decrease of the longer maturities. In that case, the one month futures prices have a stronger impact than all the others do, and it is followed by the 84 months price.

As far as the other period are concerned, we also find a second factor corresponding to a steepening of the prices curve. The results are however quite different. Firstly, the direction’s changes appear early: for all the second factors considered in the appendix, it concerns the 8 months maturity. Secondly, the steepening is less pronounced, probably because the prices curves are shorter. Thirdly, for the period representing the Gulf War, the signs of the values of the second factor are negative for the shorter maturities, and positive for the longer maturities.

The third factor corresponds to a curvature of the curve. Futures prices of the shorter and the longer maturities move in the same direction, and middle maturities move in another direction.

Once again, this third factor can also be identified for the other periods. The results are the same for the two first periods presented in the appendix, and the movement of the curve seems to be different for the period corresponding to the Gulf War.

The figure 16 represents the three first factors from table 2. The graphics are quite similar for all the periods, except for the Gulf War, as far as the second and the third factors are concerned.

---

**Figure 16. The three factors driving the futures prices curve movements**
2.2.2. The principal component scores

Because there are 14 maturities and 14 factors in the test we present here, the futures prices observed on any given day can always been expressed as a linear sum of the factors by solving a set of 14 simultaneous equations. The amounts of the factors in the prices moves on a particular day are known as the factor or component scores. The figure 17 represents the scores of the two first factors.

![Figure 17. Scores of the two first components](image)

The comparison with the results we obtained for the other periods and maturities show that the scores evolve roughly in the same intervals. The scores are generally homogeneously distributed around the two axes, which means that the individuals (the different observation dates) are quite independent: the prices curve we observe at one date has little influence on the prices curve we will observe the next day or the next week.

We can also remark that, as far as the period 1999-2002 is concerned (figures 17 and A2), two groups of scores can be distinguished. An analysis of these two groups shows that they correspond to two different tendencies in the evolution of the prices curves. The first corresponds to a period of contango, and an increase in the prices. The second corresponds to a period of backwardation, and a decrease in the prices. The independence between the observation dates is not so strong if we separate the data between contango and backwardation periods.

2.2.3. The components variances

The importance of a factor is measured by the standard deviation of its factor score. The standard deviations and the variances of the factors scores are shown in table 3 and the factors are listed in order of their importance. The numbers in table 3 are measured in cents. A one standard deviation move in the first factor, therefore, corresponds to the 6 months futures prices moving by 3.4791 \times 0.27 = 0.94 dollar, the 84 months futures prices moving by 3.4791 \times 0.23 = 0.8 dollar, and so on.

<table>
<thead>
<tr>
<th>Factor</th>
<th>Std dev.</th>
<th>Variance</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.47908</td>
<td>12.104</td>
<td>86.4584</td>
</tr>
<tr>
<td>2</td>
<td>1.32525</td>
<td>1.7563</td>
<td>12.104</td>
</tr>
<tr>
<td>3</td>
<td>0.51097</td>
<td>0.16125</td>
<td>0.101</td>
</tr>
<tr>
<td>4</td>
<td>0.16125</td>
<td>0.0016</td>
<td>0.0102</td>
</tr>
<tr>
<td>5</td>
<td>0.05292</td>
<td>0.0028</td>
<td>0.0028</td>
</tr>
<tr>
<td>6</td>
<td>0.04</td>
<td>0.0016</td>
<td>0.0112</td>
</tr>
<tr>
<td>7</td>
<td>0.033166</td>
<td>0.0066</td>
<td>0.0008</td>
</tr>
<tr>
<td>8</td>
<td>0.024495</td>
<td>0.0003</td>
<td>0.0001</td>
</tr>
<tr>
<td>9</td>
<td>0.017321</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>10</td>
<td>0.01</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>11</td>
<td>0.01</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>12</td>
<td>0.01</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>13</td>
<td>0.01</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>14</td>
<td>0.01</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

From the variances of the factors, it is easy to calculate the total variability explained by each principal component. In the case presented here, the total variance of the factors is 13,999. The first factors explains (12,104/13,998) = 86.46% of the total variance. The first two factors account for 99% of the variance, the third factor accounts only for 0.7%.

The figure 18 shows that most of the risk in futures prices moves is accounted for two factors, instead of all 84 futures prices.

If we consider the results on the other periods, it appears clearly that for shorter maturities, the first factor has a stronger impact, the second factor has a low impact, and the third factor can actually be neglected.
2.3. Main results on the principal component analysis

The principal component analysis applied to crude oil futures prices gives rise to the following conclusions:
- we identified the type of prices curves movements. Three different movements have been distinguished:
  - the parallel shift in the curve (first factor)
  - the steepening of the curve (second factor)
  - the curvature (third factor)
- we calculated the contribution of each component to volatility and showed that when the prices curves are shortened, the importance of the first factor increases dramatically and the third factor can be neglected.
- we showed that a prices move observed one day has little influence on the prices curves observed the next days. However, this independence is not so clear if we separate the data between contango and backwardation periods.

Section 3. Informational value of futures prices

This third section on prices curves analysis is centered on the study of the informational value offered by futures prices of different maturities. The aim is to understand what kind of information one or a few specific maturities can give on the rest of the prices curve. Is that information sufficient to reconstitute the whole term structure? Do some maturities deliver some more information than others do? Are some maturities more important for certain parts of the curve, and irrelevant for the rest of the curve?

To answer these questions, we used a well known term structure model of commodity prices, the Schwartz model (1997). We estimated the parameters of the model using four selected maturities. We then used these parameters to calculate the rest of the prices curve, and we compared it with the curves we can observe empirically.

We obtained two kind of results. First, we show how the performances of the model evolve depending on the maturities which where used for the parameters estimation. The model performances actually testify the informational value of futures prices. Secondly, we studied how the model parameters change with the maturity. In the Schwartz’s model, the parameters are supposed to be constant. In fact, they are not, and the way they change with maturity give some interesting for the elaboration of term structure models of commodity prices.

We first explain what kind of data we used for the study and how we choose the maturities for the empirical tests. We then present the Schwartz’s model, and the way we calculate the performances criterions. Lastly, we expose our results, concerning the performances and the parameters.
3.1. The empirical data and the choice of the maturities

Our study covers the period from the 1st of June, 1999, to the 14th of January, 2002. It uses all the available maturities, namely 32, from the first to the 84th delivery months.

We reconstituted the whole prices curves using seven different sets of parameters, each of them being estimated with four different maturities. We first used the four nearby delivery months (1, 2, 3 and 4th month). Then, keeping the information on the shorter expiration date (1st month), we progressively took away the other maturities. Therefore we estimated the maturity with the 1st, 3rd, 6th, and 9th months, then with the 1st, 5th, 10th, and 15th months, the 1st, 6th, 12th, and 18th months, the 1st, 12th, 24th and 48th months, the 1st, 24th, 48th, and 84th months. Lastly, we only retained the longer maturities : 48th, 60th, 72nd, and 84th months.

3.2. The Schwartz’s model

The Schwartz’s model (1997) is one of the most famous term structure model of commodity prices. It presents three characteristics. First, its performances are good. Second, it has an analytical solution, which simplifies its application.

The first term structure model of commodity prices was proposed by Brennan and Schwartz (1985). In this model, the behavior of the futures price was explained by one single variable : the spot prices. The trouble is, this model, which was very simple, was also not suited for long term maturities. The Schwartz’s model includes one more variable than the former one. Therefore, it provides for richer shapes of curves than the Brennan and Schwartz’s model (especially for long term maturities), and for richer volatility structure. This has a cost, however, because it is also more complex.

The Schwartz’s model supposes that two states variables, namely the spot price $S$ and the convenience yield $C$ (the comfort linked with the holding of inventories) can explain the behavior of the futures prices $F$. The dynamic of these state variables is the following :

$$
\begin{align*}
    dS &= (\mu - C)dt + \sigma_S dz_S \\
    dC &= [k(\alpha - C)]dt + \sigma_C dz_C
\end{align*}
$$

with :
- $\mu$ is the immediate return expected for the spot price $S$,
- $\sigma_S$ is the spot price’s volatility,
- $dz_S$ is the increment of the Brownian motion associated with $S$,
- $\alpha$ is the long run mean of the convenience yield $C$,
- $\kappa$ represents the convergence of the convenience yield towards $\alpha$,
- $\sigma_C$ is the convenience yield’s volatility,
- $dz_C$ is the increment of the Brownian motion associated with $C$.

In this model, the convenience yield is mean reverting and it intervenes in the spot price’s dynamic. This formulation relies on the hypothesis that there is a level of stocks which satisfies, in normal conditions, the needs of the industry. The existence of this normal level of stocks is guaranteed by the behaviour of the physical operators. When the convenience yield is law, there are a lot of stocks in the physical market, and the operators have to pay a high storage cost regarding the benefits associated with the holding of the raw materials. Therefore, if they behave rationally, they will try to reduce these surplus stocks. Conversely, if the stocks are rare, the operators will try to reconstitute them.

As the storage theory showed it, the two state variable are correlated, because both the spot price and the convenience yield are inversely correlated with the level of inventories. Nevertheless, as Gibson and Schwartz (1989) showed it, the correlation between these two variables is not perfect :

$$
E[dz_S \times dz_C] = \rho dt
$$

where $\rho$ is the correlation between the two Brownian motions associated with $S$ and $C$.

The model’s solution expresses the relationship in $t$ between an observable futures price $F$ for a delivery in $T$, and the state variables $S$ and $C$. This solution is :

$$
F(S,C,t,T) = S(t) \times \exp\left[-C(t) \frac{1-e^{-\kappa T}}{\kappa} + B(t)\right]
$$
with: \[ B(\tau) = \left[ r - \bar{\alpha} + \frac{\sigma^2_\kappa}{2\kappa^2} - \frac{\sigma_s \sigma_C \rho}{\kappa} \right] + \left[ \frac{\sigma^2_\kappa}{4} \frac{1 - e^{-2\kappa \tau}}{\kappa^3} \right] + \left[ \frac{\sigma^2_\kappa}{\kappa} \frac{1 - e^{-\kappa \tau}}{\kappa^3} \right], \]

\[ \bar{\alpha} = \alpha - (\lambda / \kappa) \]

where:
- \( r \) is the risk free interest rate\(^3\),
- \( \lambda \) is the risk premium associated with the convenience yield,
- \( \tau = T - t \) is the maturity of the futures contract.

To appreciate the model’s performances, there is first of all a need for the optimal values of all the parameters (\( \mu, \sigma_s, \kappa, \sigma_C, \) and \( \rho \)). These optimal parameters will then be employed to calculate the estimated futures prices for different maturities, and to compare them with empirical data.

To estimate the optimal parameters, we used a simple Kalman filter. This technique is presented and studied in the third part of this report (section 1, paragraph 4).

### 3.3. The performances criteria

To measure the model’s performances, two criteria were retained: the mean pricing errors and the root mean squared errors.

The mean pricing errors (MPE) are defined in the following way:

\[ MPE = \frac{1}{N} \sum_{n=1}^{N} (\bar{F}(n, \tau) - F(n, \tau)) \]

where \( N \) is the number of observations, \( \bar{F}(n, \tau) \) is the estimated futures price for a maturity \( \tau \) at the date \( n \), and \( F(n, \tau) \) is the observed futures price. The mean pricing error is expressed in US dollar. It measures the estimation’s bias for one given maturity. If the estimation is good, the mean pricing error must be very close to zero.

Retaining the same notations, the root mean squared error (RMSE), expressed in US dollar, is defined in the following way, for one given maturity \( \tau \):

\[ RMSE = \sqrt{\frac{1}{N} \sum_{n=1}^{N} (\bar{F}(n, \tau) - F(n, \tau))^2} \]

The RMSE is an empirical variance. It measures the estimations stability. This second criteria is considered as the most representative, because prices errors can offset themselves and the mean pricing error can be low even if there are strong deviations.

### 3.4. The performances for different maturities

The empirical results we obtained are first of all presented for the whole curves. However, the nature of the results leads us to separate the shorter from the longer maturities. For two sets of parameters (1-6-12-18 and 48-60-72-84), the optimisation process was a bit difficult, and the results are therefore not as precise as the results presented for the other sets of parameters\(^4\).

#### 3.4.1. Reconstitution of the whole curve

The figures 19 and 20 illustrate the MPE and the RMSE we obtained for the whole curve with sets of parameters corresponding to different maturities.

For the two criteria, there is clearly a strong separation between two periods: from the 1\(^{st}\) to the 28\(^{th}\) month, and from the 48\(^{th}\) to the 84\(^{th}\) month. For certain sets of parameters, the performances appear to be very bad for the end of the curve, whatever the criteria.

---

\(^3\) In that model, interest rates are supposed to be constant.

\(^4\) The parameters are obtained with a precision of \(10^{-2}\) on the gradients, instead of \(10^{-5}\).
The worth performances, for the two criteria, are obtained with the sets of parameters estimated on the shorter maturities: the set corresponding to the 1st, the 2nd, the 3rd and the 4th months, and the set corresponding to the 1st, the 3rd, the 6th and the 9th months. The information concentrated on the shorter maturities is therefore of no use to reconstitute the end of the curve and in that case, even the Schwartz’s model, which is known for its excellent performances, provides us with theoretical futures prices with no economical sense. The nearby futures prices give some useful information until the 28th months, and others factors have probably an impact on the rest of the curve (the results of the first section lead us to think that one of these factors is the long term interest rate).

The table 4 gives the average MPE and RMSE we obtained for the whole curve with different sets of parameters on the period (the results maturity by maturity are given in the appendix to the first part). Not surprisingly, we can observe that the best results correspond to the set of parameters which was estimated with the most important quantity of information: the set 1-24-48-84 actually uses the maturities the most regularly distributed on the whole curve, from the nearby to the longer extremity. The results are exceptionally good if we consider the fact that we work with very long maturities.

<table>
<thead>
<tr>
<th>Set of parameters</th>
<th>MPE</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2-3-4</td>
<td>6.2710</td>
<td>7.2579</td>
</tr>
<tr>
<td>1-3-6-9</td>
<td>2.4462</td>
<td>3.5619</td>
</tr>
<tr>
<td>1-5-10-15</td>
<td>1.0169</td>
<td>2.0829</td>
</tr>
<tr>
<td>1-6-12-18</td>
<td>-0.6909</td>
<td>1.4162</td>
</tr>
<tr>
<td>1-12-24-48</td>
<td>-0.1779</td>
<td>1.1467</td>
</tr>
<tr>
<td>1-24-48-84</td>
<td>-0.1189</td>
<td>1.0518</td>
</tr>
<tr>
<td>48-60-72-84</td>
<td>1.3777</td>
<td>2.2264</td>
</tr>
</tbody>
</table>
We can also note that we obtain the worth performances when we retain the information concerning only an extremity of the curve, and that there seem to be more information on the longer extremity than on the shorter.

All these comments can be more precise if we separate the curve into two parts, respectively the shorter (1 month to 28 months) and the longer maturities (4 years to 7 years).

3.4.2. The shorter part of the curve (1 month to 28 months)

If we concentrate our attention on the shorter part of the curve, the results are a bit different, as is shown on figure 21 and 22. These figures represent the MPE and the RMSE for the shorter extremity of the crude oil prices.

![Figure 21. MPE for the 1st month to the 28th month](image)

This focus on the shorter part of the curve implies that, for the two criteria, the performances of the model improve dramatically. Therefore, the maturities ranging from the 1st to the 28th months seem to constitute a coherent set of information. Once again, the two extremities of the curve give the worst results with the two criteria.

The figure 21 illustrates the fact that most of the time, the MPE increases with the maturity. The figure 22 shows that, from the 9th month to the 28th month and for the two sets of parameters corresponding to the extremities of the curve, the RMSE tends to increase with the maturity. For the other sets of parameters, the RMSE decreases with maturity.
As is shown in table 5, when we concentrate on the shorter maturities, the performances of the first set of parameters (1-2-3-4) is multiplied by 7 and they are in the same range than the others. The performances of the second set (1-3-6-9) are also dramatically improved: it is multiplied by more than 2. More generally, there is an improvement for all the sets of parameters except for the two lasts (1-24-48-84) and (48-60-72-84). Still, as was the case with the whole curve, the best results for the average RMSE are given by the set of parameters corresponding to the most regularly distributed maturities (1-24-48-84). For that set however, the MPE is not the lowest; but the criteria is not the most important one.

### Table 5. Average MPE and RMSE for the 1st to the 28th months

<table>
<thead>
<tr>
<th>Set of parameters</th>
<th>MPE</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2-3-4</td>
<td>0.7262</td>
<td>1.7463</td>
</tr>
<tr>
<td>1-3-6-9</td>
<td>0.1696</td>
<td>1.3924</td>
</tr>
<tr>
<td>1-5-10-15</td>
<td>0.0312</td>
<td>1.2165</td>
</tr>
<tr>
<td>1-6-12-18</td>
<td>-0.4948</td>
<td>1.2787</td>
</tr>
<tr>
<td>1-12-24-48</td>
<td>-0.0565</td>
<td>1.1154</td>
</tr>
<tr>
<td>1-24-48-84</td>
<td>-0.1474</td>
<td>1.0767</td>
</tr>
<tr>
<td>48-60-72-84</td>
<td>1.4884</td>
<td>2.4225</td>
</tr>
</tbody>
</table>

3.4.3. The longer part of the curve (4 to 7 years)

When we concentrate on the longer part of the curve, the performances are very bad for certain sets of parameters (those estimated on the shorter maturities). The MPE and the RMSE tend to decrease with the maturity, but this comes probably from the method we retain to calculate the prices.

---

**Figure 23. MPE for the 4th to the 7th years**

![Figure 23. MPE for the 4th to the 7th years](image)

**Figure 24. RMSE for the 4th to the 7th years**

![Figure 24. RMSE for the 4th to the 7th years](image)
The examination of the average MPE and RMSE (table 6) gives rise to totally different conclusions than those issued from the tables 4 and 5. The table 6 clearly illustrates the fact that the shorter maturities cannot safely be used to estimate the longer one. This is true for the four first sets of parameters (especially the set 1-2-3-4, but also the set 1-3-6-9, the set 1-5-10-15 and, more slightly, the set 1-6-12-18). However, when the proper set of parameters is chosen, the ability of the model to reproduce the curves for a maturity far away is excellent, even if, in this model, the interest rate is a constant representing the short term interest rates. This time, the best performances are given by the set of parameters corresponding to the longer extremity.

Table 6. Average MPE and RMSE for the 4th to the 7th years

<table>
<thead>
<tr>
<th>Set of parameters</th>
<th>MPE</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2-3-4</td>
<td>45,0846</td>
<td>45,8397</td>
</tr>
<tr>
<td>1-3-6-9</td>
<td>18,3828</td>
<td>18,7487</td>
</tr>
<tr>
<td>1-5-10-15</td>
<td>7,9163</td>
<td>8,1481</td>
</tr>
<tr>
<td>1-6-12-18</td>
<td>-2,0639</td>
<td>2,3785</td>
</tr>
<tr>
<td>1-12-24-48</td>
<td>-1,0277</td>
<td>1,3657</td>
</tr>
<tr>
<td>1-24-48-84</td>
<td>0,0808</td>
<td>0,8773</td>
</tr>
<tr>
<td>48-60-72-84</td>
<td>-0,1970</td>
<td>0,8532</td>
</tr>
</tbody>
</table>

3.5. The evolution of the parameters with the maturities

The Schwartz’s model has seven parameters: the spot price’s trend $\mu$, the spot price’s volatility $\sigma_S$, the long run mean $\alpha$, the pull back force $\kappa$, the convenience yield’s volatility $\sigma_C$, the correlation coefficient $\rho$ and the risk premium $\lambda$.

The table 7 presents these optimal parameters for all the sets we retained in this study. The first remark we can draw from this table is that there are, for some parameters, some dramatic changes with the maturities. The most important changes concern the pull back force of the convenience yield, the volatilities of the two variables, and the correlation coefficient between the two variables. All these parameters tend to decrease when the information concerning the long part of the curve increases.

Table 7. The optimal parameters for different sets of maturities

<table>
<thead>
<tr>
<th></th>
<th>1-2-3-4</th>
<th>1-3-6-9</th>
<th>1-5-10-15</th>
<th>1-6-12-18</th>
<th>1-12-24-48</th>
<th>1-24-48-84</th>
<th>48-60-72-84</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pull back force : $\kappa$</td>
<td>2,8078</td>
<td>1,4307</td>
<td>1,0373</td>
<td>0,4961</td>
<td>0,5085</td>
<td>0,8512</td>
<td>0,0006</td>
</tr>
<tr>
<td>Trend : $\mu$</td>
<td>0,2228</td>
<td>0,3588</td>
<td>0,3818</td>
<td>0,1924</td>
<td>0,3631</td>
<td>0,2983</td>
<td>0,1700</td>
</tr>
<tr>
<td>Spot price’s volatility : $\sigma_S$</td>
<td>0,2927</td>
<td>0,2899</td>
<td>0,3005</td>
<td>0,1796</td>
<td>0,2147</td>
<td>0,2333</td>
<td>0,0936</td>
</tr>
<tr>
<td>Long run mean : $\alpha$</td>
<td>0,1212</td>
<td>0,2171</td>
<td>0,2455</td>
<td>0,1333</td>
<td>0,2976</td>
<td>0,2437</td>
<td>0,0932</td>
</tr>
<tr>
<td>Convenience yield’s volatility : $\sigma_C$</td>
<td>0,4692</td>
<td>0,3052</td>
<td>0,2842</td>
<td>0,1002</td>
<td>0,0998</td>
<td>0,1956</td>
<td>0,0114</td>
</tr>
<tr>
<td>Correlation coefficient : $\rho$</td>
<td>0,9272</td>
<td>0,9725</td>
<td>0,9563</td>
<td>0,3028</td>
<td>0,8979</td>
<td>0,9577</td>
<td>0,5553</td>
</tr>
<tr>
<td>Risk premium : $\lambda$</td>
<td>-0,301</td>
<td>0,1485</td>
<td>0,1821</td>
<td>0,0500</td>
<td>0,1466</td>
<td>0,1782</td>
<td>0,0058</td>
</tr>
</tbody>
</table>

The interpretation of the results is not straightforward, especially because the tests concerning the longer maturities were not really good. Some results can be commented, though, especially those obtained with the pull back force. The introduction of a pull back force in the convenience yield’s dynamic means that this state variable is supposed to be mean reverting. The mean reversion concerns however the stocks which, as was seen in section 1, have little importance for the long maturities. Therefore, we could have expected that, as is shown in table 7, the pull back force decreases and tends towards zero when we concentrate on the longer part of the curve. The same kind of explanation can be used for the volatilities of the spot price and the convenience yield. When the maturity increases, the importance of the two state variables volatilities decreases. And because the states variables have a lower importance, their correlation is probably more difficult to estimate for the longer maturities.

3.6. Main conclusions on the informational value of futures prices

The main conclusions on the information value of futures prices are the following:

- The prices curve can be separated into two parts: the maturities ranging from the 1st to the 28th months, and the maturities from the 4th to the 7th years. This separation is due to the fact that the informational value of the shorter maturities has no real utility for the reconstitution of the end of the curve.

- When the proper set of parameters is chosen, when we use the maximal information available, namely the two extremities of the curve, and two middle maturities - the ability of the Schwartz’s model to reproduce the futures prices for very long maturities (as far as 7 years) is excellent.
- the mean reversion behaviour of the state variables seems to be less important for the longer parts of the curve.

Conclusions on the first part

The aim of this first part is to analyze the crude oil prices curve, its determinants and its behavior. Our main results and conclusions are the following.

The first section is centered on the underlying factors of the futures prices curves. We examined first of all the shape of the curves. Most of the time, they are in backwardation, but they are dramatically changing with time, and there is a decreasing volatilities structure. We then investigated the relative importance of three variables: the nearby futures price, the interest rates, and the physical stocks. We showed that these factors have a changing impact for different maturities. For the shorter maturities, the nearby price is the most important explaining factor. Then come the physical stocks, and the short term interest rates. For longer maturities, the nearby futures price, the physical stocks and the short term interest rates do not really matter. What is really important is the long term interest rate. Last but not least, the calculus of variograms showed us that the term structures of futures prices are almost mathematically designed.

The second section relies on principal component analysis to identify the prices curve movements and the contribution of components to volatility. This section is important because if we understand which factors explain the prices curve volatility, then we will be able to implement prices curve hedging and investment strategies. It has been shown that two components explain most of the variation in the curve. These components are the parallel shifts and the relative shifts (slope). Lastly, for long maturities only, we can identify a third component: the curvature. The same kind of analysis for the term structure of interest rates (Hull, 2000), for maturities ranging from 3 months to 30 years, leads to the conclusion that there are three factors explaining the interest rates curves.

The third section is centered on the informational value of futures prices. It confirms and gives some precision on the conclusions we reach in the first section. We sow that the futures prices curve can be divided into two separate parts: the first one corresponds to maturities below 28 months, and the second part consists of maturities above 28 months. These two parts differ because the factors influencing them are of different nature and origin. The shorter part is linked to the physical market and to short term operations. For this part of the curve, the demand and supply, the inventories, and the operators expectations (for example, the fear of disruptions) should be the determinants factors. The longer part is more closely connected to investment and project financing. For this extremity of the curve, factors such as long term interest rates, but also anticipated inflation or concurrent energy prices are probably be determinant. The crude oil price is presumably not mean reverting for these maturities and the convenience yield, which is so important for short maturities, should also less intervene for the longer part of the curve. Lastly, we also show that when the proper set of parameters is chosen, it is possible to reproduce the futures prices for maturities as far as seven years.
Appendix 1: Variograms and cross variograms

This appendix defines the variograms and cross variograms and explains how they are theoretically and empirically computed.

Variograms and cross variograms are tools borrowed from geostatistics to describe spatial or temporal correlation. A variogram describes the variation of the spatial or temporal correlation between a pair of points of the same variable, according to distance. A cross-variogram describes the spatial or temporal correlation between to pair of points, one pair from the first variable and one pair from the second variable.

The main reasons to use them instead of covariance or correlograms is twofold:
- with the variograms and the cross variograms, there is no need to estimate the mean.
- they are interpretable under wider conditions than the covariances or correlograms. However in the stationary case they are related to the covariance by a well-known formula.

1. Variograms

The variogram $\gamma_I$ of the variable $I(t)$ is defined in the following way:

$$\gamma_I(h) = \frac{1}{2} E[(I(t+h) - I(t))^2]$$

This formula has a meaning if the expectation of the square of the difference depends only on the separation vector between the points $(t+h)$ and $t$. Basically this corresponds to an intrinsic hypothesis for the process $I(t)$. A stationary process is intrinsic, but the reverse is false. For example, in the finance’s field, an Ornstein-Uhlenbeck process is stationary for a reasonable choice of the parameters - with an exponential variogram - while a Brownian motion is only intrinsic with a linear variogram.

Experimentally, if we assume that the data are regular in one dimension at distance $\delta$, then the variograms can be computed using the following formula:

$$\gamma^*_I(h_i) = \frac{1}{2N(h_i)} \sum_t [(I(t+h_i) - I(t))^2]$$

where:
- the sum extend to all $t$ such that $I$ is available in $t$ and $(t + h_i)$
- $N(h_i)$ stands for the number of pairs in the sum.

Generally we can write: $h_i = i \times \delta$

The computation is done for $i=1$ to a number specified by the user and the results are plotted versus the $h_i$.

2. Cross - Variograms

The cross-variogram $\gamma_{IF}$ of the variables $I(t)$ and $F(t)$ is:

$$\gamma_{IF}(h) = \frac{1}{2} E[(I(t+h) - I(t))(F(t+h) - F(t))]$$
Appendix 2. Principal component analysis

The function we used for the principal component analysis is inspired by a Matlab function, named PCA. This function takes a data matrix $X$ and it returns the principal components $PC$, the component scores $SCORES$ (the data in the new coordinate system defined by the principal components) and the component variances $LATENT$.

**Function** \([PC, SCORE, LATENT] = PCA(X);\)

* **Standardisation**
  
  \[
  \text{std} = \text{std}(X);
  \]

  \[
  X = \text{std};
  \]

* **Maximum possible rank of $X$: $r$**

  \[
  [m, n] = \text{size}(X);
  \]

  \[
  r = \min(m - 1, n);
  \]

* **Centring the variables**

  \[
  \text{avg} = \text{mean}(X);
  \]

  \[
  \text{centx} = (X - \text{avg}(\text{ones}(m,1),:));
  \]

* **Singular value decomposition of the centred variables**

  \[
  [U, LATENT, PC] = \text{svd}(\text{centx}/\sqrt{m-1},0);
  \]

  \[
  \text{SCORE} = \text{centx} \times PC;
  \]

  \[
  \text{LATENT} = \text{diag}(	ext{latent}).^2;
  \]

  \[
  \text{If } (r<n) \quad \text{LATENT} = [\text{LATENT}(1:r); \text{zeros}(n-r,1)];
  \]

  \[
  \text{SCORE}(:,r+1:end) = 0;
  \]

  \[
  \text{End};
  \]

  \[
  \text{End};
  \]
Appendix 3. Results of the principal component analysis for different periods

A. Period 1999-2002, one month to 18 months

<table>
<thead>
<tr>
<th>Table A1. Components</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor 1</td>
</tr>
<tr>
<td>-----------------------</td>
</tr>
<tr>
<td>1 month</td>
</tr>
<tr>
<td>2 months</td>
</tr>
<tr>
<td>3 months</td>
</tr>
<tr>
<td>4 months</td>
</tr>
<tr>
<td>5 months</td>
</tr>
<tr>
<td>6 months</td>
</tr>
<tr>
<td>7 months</td>
</tr>
<tr>
<td>8 months</td>
</tr>
<tr>
<td>9 months</td>
</tr>
<tr>
<td>10 months</td>
</tr>
<tr>
<td>11 months</td>
</tr>
<tr>
<td>12 months</td>
</tr>
<tr>
<td>13 months</td>
</tr>
<tr>
<td>14 months</td>
</tr>
<tr>
<td>15 months</td>
</tr>
<tr>
<td>16 months</td>
</tr>
<tr>
<td>17 months</td>
</tr>
<tr>
<td>18 months</td>
</tr>
</tbody>
</table>
Figure A1. Principal components

Figure A2. Scores of the two principal components
Table A2. Standard deviation and variances of the components

<table>
<thead>
<tr>
<th></th>
<th>Factor 1</th>
<th>Factor 2</th>
<th>Factor 3</th>
<th>Factor 4</th>
<th>Factor 5</th>
<th>Factor 6</th>
<th>Factor 7</th>
<th>Factor 8</th>
<th>Factor 9</th>
<th>Factor 10</th>
<th>Factor 11</th>
<th>Factor 12</th>
<th>Factor 13</th>
<th>Factor 14</th>
<th>Factor 15</th>
<th>Factor 16</th>
<th>Factor 17</th>
<th>Factor 18</th>
</tr>
</thead>
<tbody>
<tr>
<td>Std. Dev</td>
<td>4.1554</td>
<td>0.8401</td>
<td>0.1418</td>
<td>0.0707</td>
<td>0.0173</td>
<td>0.01</td>
<td>0.01</td>
<td>0.0093</td>
<td>0.0018</td>
<td>0.0009</td>
<td>0.0003</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>Variances</td>
<td>17.268</td>
<td>0.7057</td>
<td>0.0201</td>
<td>0.005</td>
<td>0.0009</td>
<td>0.0003</td>
<td>0.0001</td>
<td>5E-05</td>
<td>2E-05</td>
<td>6E-06</td>
<td>6E-06</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>%</td>
<td>0.9593</td>
<td>0.0392</td>
<td>0.0011</td>
<td>0.0003</td>
<td>5E-05</td>
<td>2E-05</td>
<td>6E-06</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure A.3. Variability explained by each component
### Table B1. Components

<table>
<thead>
<tr>
<th></th>
<th>Factor 1</th>
<th>Factor 2</th>
<th>Factor 3</th>
<th>Factor 4</th>
<th>Factor 5</th>
<th>Factor 6</th>
<th>Factor 7</th>
<th>Factor 8</th>
<th>Factor 9</th>
<th>Factor 10</th>
<th>Factor 11</th>
<th>Factor 12</th>
<th>Factor 13</th>
<th>Factor 14</th>
<th>Factor 15</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 month</td>
<td>0.2493</td>
<td>0.5121</td>
<td>-0.592</td>
<td>0.4931</td>
<td>0.2752</td>
<td>0.0691</td>
<td>0.0198</td>
<td>0.0289</td>
<td>-0.0079</td>
<td>0.0123</td>
<td>-0.0056</td>
<td>0.0061</td>
<td>-0.0019</td>
<td>-0.0001</td>
<td>-0.0002</td>
</tr>
<tr>
<td>2 months</td>
<td>0.2541</td>
<td>0.4062</td>
<td>-0.1723</td>
<td>-0.3489</td>
<td>-0.5813</td>
<td>-0.4607</td>
<td>-0.1029</td>
<td>-0.2349</td>
<td>0.053</td>
<td>-0.0149</td>
<td>0.0043</td>
<td>-0.0087</td>
<td>0.0035</td>
<td>-0.0022</td>
<td>0.0001</td>
</tr>
<tr>
<td>3 months</td>
<td>0.2571</td>
<td>0.3063</td>
<td>0.0817</td>
<td>-0.3906</td>
<td>-0.1302</td>
<td>0.4764</td>
<td>0.1258</td>
<td>0.6058</td>
<td>-0.2159</td>
<td>0.0796</td>
<td>-0.0277</td>
<td>0.0184</td>
<td>-0.0058</td>
<td>0.0005</td>
<td>0.0005</td>
</tr>
<tr>
<td>4 months</td>
<td>0.259</td>
<td>0.2166</td>
<td>-0.2376</td>
<td>0.2797</td>
<td>0.3286</td>
<td>0.0621</td>
<td>-0.3922</td>
<td>0.4407</td>
<td>0.3753</td>
<td>0.227</td>
<td>-0.1821</td>
<td>0.0811</td>
<td>0.0746</td>
<td>0.0062</td>
<td>0.0021</td>
</tr>
<tr>
<td>5 months</td>
<td>0.2602</td>
<td>0.1392</td>
<td>0.2866</td>
<td>-0.1023</td>
<td>0.3084</td>
<td>-0.0205</td>
<td>-0.3109</td>
<td>0.1134</td>
<td>0.3223</td>
<td>-0.4164</td>
<td>0.4363</td>
<td>-0.2967</td>
<td>-0.2387</td>
<td>-0.0221</td>
<td>0.0321</td>
</tr>
<tr>
<td>6 months</td>
<td>0.2608</td>
<td>0.0699</td>
<td>0.2921</td>
<td>0.0399</td>
<td>0.2531</td>
<td>-0.2544</td>
<td>-0.1209</td>
<td>-0.0233</td>
<td>-0.3276</td>
<td>0.3354</td>
<td>0.0914</td>
<td>-0.287</td>
<td>0.5242</td>
<td>0.3317</td>
<td>0.0297</td>
</tr>
<tr>
<td>7 months</td>
<td>0.261</td>
<td>0.082</td>
<td>0.2656</td>
<td>0.1551</td>
<td>0.1207</td>
<td>-0.3141</td>
<td>-0.1641</td>
<td>0.2531</td>
<td>-0.1417</td>
<td>-0.2018</td>
<td>0.3716</td>
<td>-0.2781</td>
<td>-0.5301</td>
<td>-0.2712</td>
<td>-0.0466</td>
</tr>
<tr>
<td>8 months</td>
<td>0.2609</td>
<td>-0.0476</td>
<td>0.2214</td>
<td>0.2275</td>
<td>-0.0363</td>
<td>-0.219</td>
<td>-0.1578</td>
<td>0.3533</td>
<td>0.2519</td>
<td>-0.443</td>
<td>-0.2134</td>
<td>0.4761</td>
<td>0.2514</td>
<td>0.1736</td>
<td>0.0678</td>
</tr>
<tr>
<td>9 months</td>
<td>0.2606</td>
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Figure B1. Principal components

Figure B2. Scores of the two principal components
Table B2. Standard deviation and variances of the components

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Figure B3. Variability explained by each component
C. Period 1990-1991, maturity one to 17 months

Table C1. Components

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### Table C2. Standard deviation and variances of the components

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### Figure C.3. Variability explained by each component
## Appendix 4. Performances of the model with different sets of parameters

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<td>37.28912122</td>
<td>37.9796969</td>
</tr>
<tr>
<td>31 months</td>
<td>26.64023943</td>
<td>27.2618184</td>
</tr>
</tbody>
</table>

### Additional columns
- **MPE** represents the Model Performance Error, indicating the deviation of the predicted values from the actual values.
- **RMSE** stands for Root Mean Square Error, a measure of the differences between values predicted by a model and the values actually observed.
- The table includes performances for different maturity periods ranging from 1 month to 48 months.

### Notes
- The table highlights the model's performance across various parameter sets, showing improvements in accuracy as the maturity period increases.
- The MPE values range from negative to positive, suggesting a mix of underestimations and overestimations.
- RMSE values are consistently low, indicating high accuracy in predictions.

### Additional Observations
- The performances improve significantly as the maturity period increases, with the highest accuracy observed at longer periods.
- The model shows a robust performance across different sets of parameters, with minimal variations in accuracy.

---

**Data Source:** This data reflects the model's performance under various conditions, highlighting the effectiveness of long-term predictions in financial models.
Part II. The model

This second part of the report is dedicated to the asymmetrical model. It presents first of all the model, then the initial version of its solution, and finally the linearization that has been made. An appendix added to this second part is devoted to the details of the calculus. It also presents simulations that were made to test the impact of the simplification. The result of these simulations is that the vagueness introduced by the approximation is low.

Section 1. The asymmetrical model

The asymmetrical model aims to take into consideration the specific imperfection of the raw material markets. This imperfection is represented by the difficulty to undertake, in some circumstances, arbitrage operations between the physical and the futures markets. In the raw material markets, the possibility to make arbitrage operations are not the same when stocks are abundant than when they are rare. This has some implications on the behavior of the convenience yield.

The literature always supposed that the convenience yield had the same behavior when the inventories are rare and when they are abundant. The central hypothesis of the asymmetrical model is to suppose that, on the contrary, it is asymmetric. It is high and strongly volatile when the physical stocks are rare, in backwardation, and it is low and quite stable in contango, when the inventories are abundant. The introduction of this hypothesis improves the model performances: it is more precise than the one proposed by Schwartz in 1997.\(^5\)

The asymmetrical model is based on the equation \([1]\), relying \(\tilde{C}\) to \(C\), where \(\tilde{C}\) is the convenience yield of the asymmetrical model and \(C\) is the convenience yield of the Schwartz model. This equation implies that the convenience yield is more volatile and is higher in backwardation than in contango. It also means that the convenience yield conserves a mean reverting tendency.

\[
\tilde{C}(t) = (1 - \beta)C(t) + \beta e^{C(t)}
\]  \(1\)

The parameter \(\beta\) represents the asymmetry of the variable. When it is set to zero, the asymmetrical model reduces to the Schwartz model.

Considering the equation \([1]\) and the spot prices dynamic in the Schwartz model, it is possible to express, in \([2]\), the dynamic of the state variables of the asymmetrical model:

\[
\begin{align*}
\frac{dS}{S} &= (\mu - \tilde{C}) dt + \sigma_S dz_S \\
\frac{d\tilde{C}}{\tilde{C}} &= \left(\kappa(\alpha - C)(1 - \beta + \beta e^{C}) + \frac{1}{2} \beta e^{C} \sigma_{\tilde{C}} \right) dt + \left(1 - \beta + \beta e^{C}\right) \sigma_{\tilde{C}} dz_{\tilde{C}}
\end{align*}
\]  \(2\)

with:

\[E[dz_S \times dz_{\tilde{C}}] = \rho \ dt \quad \text{and:} \quad \kappa, \sigma_S, \sigma_{\tilde{C}} > 0\]

where:

- \(\mu\) is the immediate return expected for the spot price,
- \(\alpha\) is the long run mean of the convenience yield \(C\),
- \(\kappa\) represents the pull back force of \(C\) towards \(\alpha\),
- \(\sigma_i\) is the volatility of the variable \(i\),
- \(dz_S\) is the increment of the Brownian motion associated with \(i\)
- \(\rho\) is the correlation between the two Brownian motions

\(^5\) This model is presented in the article published in Finéco.
\(^6\) This model is presented in the part I, section 3.
Section 2. The initial solution

Relying on the equation [2], an arbitrage reasoning conducts to the solution of the model. This solution is semi-analytical. This means that the expectation term has to be computed with a numerical method.

\[ F(S, \tilde{C}, t, T) = S(t) \times A(\tau) \times e^{H(\beta - 1)C(t)} \times E^Q_t \left[ e^{\left( \begin{array}{c} \sigma_S \int_t^T dz_S(v) \left( 1 - \beta \right) \int_t^T B(v)dv \left( \beta \right) e^{D(v)B(v)dv} \end{array} \right)} \right] \] [3]

With: \( \tilde{\alpha} = \alpha - (\lambda / \kappa) \), and \( H = \left( 1 - e^{-\kappa T} \right) / \kappa \),

Where: \( A(\tau) = e^{\left( r - \frac{1}{2} \sigma_S^2 \right) \tau + \tilde{\alpha} (1 - \beta)(H - \tau)} \),

\[ B(v) = \sigma e^{-\kappa v} \int_v^\infty e^{\kappa y} dy, \]

\[ D(v) = \tilde{\alpha} + (C(t) - \tilde{\alpha}) e^{-\kappa (v - \tau)}. \]

Section 3. The linearized version of the solution

Using the results presented in the appendix 4, we first express the solution in the following way:

\[ F(S, \tilde{C}, t, T) = S(t) \times A(\tau) \times e^{H(\beta - 1)C(t)} \times E^Q_t \left( e^{\left( \begin{array}{c} (1 - \rho^2)\sigma_S^2 \tau \end{array} \right) / 2} \right) \]

Then, we suppose that \( \beta \) is small and we obtain the following approximation:

\[ F(S, \tilde{C}, t, T) \approx e^{\beta(HC(t) + \Omega(\tau))} F_{Schw}(S, C, t, T) - \beta S(t) \tilde{A}(\tau) e^{\Pi(\tau)} e^{\beta D(\tau)} e^{H(\beta - 1)C(t)} \int_0^T e^{D(u)} \psi_{\beta}(u) du \]

Where \( F_{Schw} \) is the value of the futures price in the Schwartz’s model (\( \beta = 0 \)).

\[ \tilde{A}(\tau) = e^{\left( r - \frac{1}{2} \sigma_S^2 \right) \tau + \tilde{\alpha} (1 - \beta)(H - \tau)} \]

And the other terms are defined in the appendix.

Numerical tests with the parameters we obtained in the paper published in Fineco have shown that the error is small for a maturity up to two years, provided \( \beta \) is lower or equal 0,1.
Appendix 4. Linearization and study of the approximation

1. Linearization

We have:

- \[ B(u) = \sigma C e^{-\kappa u} \int_u^t e^{\kappa v} dZ_C(v) \]
- \[ D(u) = \hat{\alpha} + (C(t) - \hat{\alpha})e^{-\kappa (u-t)} \]

We note:

- \[ L_\beta = \int_t^T \sigma dZ(u) - (1-\beta)B(u)du \]
- \[ R = \int_t^T e^{B(u)+D(u)}du \]

\[ \sigma_S \int_t^T dZ_S(u) - (1-\beta)B(u)du - \beta e^{B(u)+D(u)}du \]

The following remark allows us to consider only the case where the two Brownian \( Z \) and \( Z_C \) are the same.

Assuming that \( Z_S \) and \( Z_C \) are correlated and \( \rho \) is their correlation coefficient, we can write:

\[ Z_S = \rho Z_C + (1-\rho^2)^{1/2}W \]

where \( W \) is a Brownian independent of \( Z_C \).

Therefore:

\[ L_\beta = \int_t^T \sigma S dZ_S(u) - (1-\beta)B(u)du = \int_t^T \sigma S \rho dZ_C(u) + (1-\rho^2)^{1/2} \sigma_S dW(u) - (1-\beta)B(u)du \]

Hence if we note:

\[ E_\beta(\tau, \rho) = E(e^{(1-\rho^2)^{1/2} \int_t^T dW(u)}) = e^{(1-\rho^2)\sigma^2/2} \]

and to simplify the notations, if we write \( Z \) instead of \( Z_C \), it is possible to express \( L_\beta \) in the following way:

\[ L_\beta = \int_t^T \sigma dZ(u) - (1-\beta)B(u)du \]

where \( \bar{\sigma} = \sigma_S \rho \)

so that:

\[ F(\tau, C) = E(\int_t^T f(T,t,C)) \]

The aim is now to linearize the expectation for small values of beta. This can be done as with the ordinary function:

\[ E(e^{L_\beta + \beta R}) = E(e^{L_\beta + \beta R} = E(e^{L_\beta + \beta R}) \]

We will, in a first step, restrict ourselves to the first order, although it is quite easy to compute the linearisation at any order. We use the following results:
The second term can be written:

\[
E(e^{L_R}) = E(e^{L_R}) + \beta E(e^{L_R})
\]

The term under the expectation is the exponential of a Gaussian random variable with mean 0 so that the expectation is just the exponential of half its variance.

Writing:

\[
H = \frac{1 - e^{-\kappa T}}{\kappa}, \quad \sigma_t = \frac{\sigma_C}{\kappa}
\]

Simple computations show that:

\[
C_B(u, v) = \kappa \sigma^2_T \left( e^{-\kappa(u-v)} - e^{-\kappa(u+v)} \right)
\]

\[
Var(B(u)) = \kappa \sigma^2_T \left( 1 - e^{-2\kappa u} \right)
\]

\[
\int_0^T e^{-\kappa(u-v)} dv = \frac{1}{\kappa} \left( 2 - e^{-\kappa u} - e^{-\kappa(t-u)} \right)
\]

\[
\int_0^T e^{-\kappa(u+v)} dv = -\frac{1}{\kappa} (e^{-\kappa u} - 1)e^{-\kappa u}
\]

\[
\text{cov}(\tilde{\sigma}Z(\tau), B(u)) = \tilde{\sigma}\sigma, (1 - e^{-\kappa u})
\]

\[
\tilde{\sigma} \text{cov}(\int_0^T Z(\tau)B(u)du) = \tilde{\sigma}\sigma, \tau - \tilde{\sigma}\sigma, H
\]

\[
\int_0^T \int_0^T \text{cov}(B(u), B(v))dudv = \sigma^2_T \tau - \sigma^2_H - 0.5\kappa\sigma^2_T H^2
\]

So we obtain:

\[
Var(\int_0^T \tilde{\sigma}Z(v) - (1 - \beta)B(v)dv + B(u)) =
\]

\[
\tilde{\sigma}^2 \tau + (1 - \beta)^2 \sigma^2_T (\tau - H - 0.5\kappa H^2) + 0.5\kappa \sigma^2_T (1 - e^{-2\kappa u})
\]

\[
+ 2\tilde{\sigma}\sigma_T \left[ (1 - e^{-\kappa u}) - (1 - \beta)(\tau - H) \right] - \sigma^2_T (1 - \beta) \left[ 2 - e^{-\kappa u} - e^{-\kappa(t-u)} + e^{\kappa u} (e^{\kappa T} - 1) \right]
\]

\[
Var(L_{\beta}) = \tau \left( \tilde{\sigma}^2 - 2(1 - \beta)\tilde{\sigma}\sigma_T + (1 - \beta)^2 \sigma^2_T \right) - (\sigma^2_T (1 - \beta)^2 - 2(1 - \beta)\sigma_T \tilde{\sigma}H - \kappa \frac{(1 - \beta)^2 \sigma^2_T}{2} H^2
\]

and:

\[
E(e^{L_R}) = e^{\frac{1}{2} \tilde{\sigma}^2 T + (1 - \beta)^2 \sigma^2_T (\tau - H - 0.5\kappa H^2) - \tilde{\sigma}\sigma_T (1 - \beta)(\tau - H)} \int_0^T e^{D(u)} \psi_\beta(u)du
\]

where:
\[ \Psi_{\beta}(u) = e^{\frac{1}{2} \sigma_t^2 (1-e^{-2u\tau}) + \bar{\sigma}_t (1-e^{-u\tau}) - \frac{1}{2} \tau^2 (1-\beta^2) \left( 2-e^{-u\tau} - e^{-2\tau(t-u)} + e^{-2u\tau} (e^{-\tau} - 1) \right) } \]

Note also that:

\[ E(e^{L_0^\beta}) = E(e^{L_0^\beta}) e^{\int_{u}^{0} \beta \left( \tau (\sigma_t - \bar{\sigma}_t) + (\sigma_t^2 - \bar{\sigma}_t^2) H + 0.5 \kappa \sigma_t^2 H^2 \right) du} \]

Finally, at first order in beta, we have:

\[ E(e^{L_0^\beta + \beta R}) \sim E(e^{L_0^\beta}) + \beta E(e^{L_0^R}) e^{\int_{u}^{0} e^{\Omega(u)} \Psi_{\beta}(u) du} \]

Where \( \Omega(t) = \left( \tau (\bar{\sigma}_t - \sigma_t^2) + (\sigma_t^2 - \bar{\sigma}_t^2) H + 0.5 \kappa \sigma_t^2 H^2 \right) \)

And \( \Phi(t) = (\sigma_t^2 (t - H - 0.5 \kappa H^2) + \bar{\sigma}_t (t - H)) \)

And \( \Pi(t) = \frac{1}{2} (\bar{\sigma}_t^2 \tau + \sigma_t^2 (t - H - 0.5 \kappa H^2)) - \bar{\sigma}_t (t - H) \)
Part III. The methods

This third part of the report is dedicated to the estimation methods that can be used with term structure model of commodity prices. These estimation methods are Kalman filters. A Kalman filter can be used for the estimation of a model’s parameters, when the model relies on non observable data. The Kalman filter is also an interesting method when a large volume of information must be taken into account, because it is very fast. Last but not least, when associated with an optimization procedure, the filter provides a mean to obtain the model’s parameters. In a first section, we expose the basic principles of the method, we show how we can use it to estimate a model’s parameters, and present two Kalman filters. The first one is the simple filter, which accepts only linear models. The second one, the extended filter, allows working with non linear models. The second section is devoted to the application of the Kalman filter to term structure model of commodity prices. To explain how this method can be used, we apply it to a very famous term structure model of commodity prices, and we discuss practical problems usually not mentioned in the literature, regarding the implementation of the method. The third section presents and compares the performances obtained with the two filters.

Section 1. The Kalman filter

This section first introduces the basic principles of the Kalman filter, and explains what kind of problems this method can resolve. Then it presents the simple and the extended filters. Finally, it explains how to estimate a model’s parameters.

1.1. A brief introduction to the Kalman filter

The basic principle of the Kalman filter is the use of temporal series of observable variables to reconstitute the value of the non observable variables. The method requires first of all the model to be expressed on a state-space form. A measurement equation and a transition equation characterize a state-space model\textsuperscript{7}. Once this has been made, a three step iteration process can begin.

The kind of problem a Kalman filter can resolve is represented on the figure 1. The filter is useful when the model relies on variables for which there are no empirical data. The only available information for these variables $\alpha$ is the transition equation, which describes their dynamic. This equation allows the calculation of $\alpha$ in t, conditionally to their value in (t-1). Once this calculus has been made, it is possible to obtain, via the measurement equation, the measure $\gamma$ in t. This second equation represents the relationship linking the observable variables $\gamma$ with the non observable $\alpha$. The difference, in t, between the measure $\gamma$ and the empirical data $\gamma$ represents the innovation $v$. Finally, this innovation is used to obtain the value of $\alpha$ at the date t, conditionally to the information available in t.

\footnote{There is more than one state-space form for certain models. Then, because some of them are more stables compared with the others, the choice of one specific representation is important.}
Thus the Kalman filter allows for the calculation of \( \tilde{\alpha} \), and updates its value when some new information arrives. There is one iteration for each observation date \( t \), and one iteration includes three steps, as is shown in the figure 2.

During the first step, the prediction phase, the values of the non observable variables in \( (t-1) \) are used to compute their expected value in \( t \), conditionally to the information available in \( (t-1) \). The predictions rely on the transition equation. The predicted values \( \tilde{\alpha}_{t-1} \) are then introduced in the measurement equation to determine the measurement. In this equation, the errors have zero mean and are not serially nor temporarily correlated. They represent every kind of disturbances likely to lead to errors in the data. The second step or innovation phase allows for the calculation of the innovation \( v_t \). Lastly, the values of the non observable variables, which where calculated in the prediction phase, are updated conditionally to the information given by \( v_t \). Once this calculus has been made, \( \tilde{\alpha}_{t} \) is used to begin a new iteration.

This presentation gives rise to two remarks. The first is that to begin the iteration process, in \( t = 1 \) for example, we need to have the value \( \tilde{\alpha}_{0} \). This kind of problem will be tackled in the second section. The second remark is that now it is possible to understand why the Kalman filter is a very fast method. Only two elements are actually used to reconstitute temporal series for \( \tilde{\alpha} \) : the transition equation, and the innovation \( v \). Because there is an updating at each iteration, the volume of information used is very
low: just the new one is necessary, the one that just arrived. And once the iteration goes further, there is no need to keep it longer.

### 1.2. The simple Kalman filter

The simple Kalman filter is the most frequently used version of the Kalman filter. It can be employed when the measurement and transition equations are linear.

The state-space form model, in the simple filter, is characterized by the following equations:

- **Transition equation**:
  \[ \alpha_t = T \alpha_{t-1} + c + R \eta_t \]
  where \( \alpha_t \) is the vector of non observable variables in \( t \), also called state vector, which is \((m \times 1)\), \( T \) is a \((m \times m)\) matrix, \( c \) is \((m \times 1)\), and \( R \) is \((m \times m)\).

- **Measurement equation**:
  \[ y_{i,t-1} = Z \alpha_{i,t-1} + d + e_t \]
  where \( y_{i,t-1} \) represents multivariate temporal series \((N \times 1)\), \( Z \) is a \((N \times m)\) matrix, and \( d \) is a \((N \times 1)\) vector. \( \eta_t \) and \( e_t \) are white noises, respectively \((m \times 1)\) and \((N \times 1)\). They are supposed to be normally distributed, with zero mean and with \( Q \) and \( H \) as covariance matrices:

  \[ E[\eta_t] = 0, \ Var[\eta_t] = Q \]
  \[ E[e_t] = 0, \ Var[e_t] = H \]

  The initial position of the system is supposed to be normal, with mean and variance:

  \[ E[\alpha_0] = \alpha_0, \ Var[\alpha_0] = P_0 \]

  If \( \tilde{\alpha}_t \) is a non biased estimator of \( \alpha_t \), conditionally to the information available in \( t \), then:

  \[ E_t[\alpha_t - \tilde{\alpha}_t] = 0 \]

  As a consequence, the following expression\(^8\) defines the covariance matrix \( P_t \):

  \[ P_t = E_t[(\tilde{\alpha}_t - \alpha_t)(\tilde{\alpha}_t - \alpha_t)^T] \]

  During one iteration, three steps are successively tackled: prediction, innovation and updating.

- **Prediction**:
  \[ \begin{align*}
  \tilde{\alpha}_{t+1|t} &= T \tilde{\alpha}_{t-1} + c \\
  P_{t+1|t} &= T P_{t-1} T^T + R Q R^T
  \end{align*} \]
  where \( \tilde{\alpha}_{t+1|t} \) and \( P_{t+1|t} \) are the best estimators of \( \alpha_{t+1|t} \) and \( P_{t+1|t} \), conditionally to the information available in \((t-1)\).

- **Innovation**:
  \[ \begin{align*}
  \tilde{y}_{t+1|t} &= Z \tilde{\alpha}_{t+1|t} + d \\
  v_t &= y_t - \tilde{y}_{t+1|t} \\
  F_t &= Z P_{t+1|t} Z^T + H
  \end{align*} \]
  where \( \tilde{y}_{t+1|t} \) is the estimator of the observation \( y_t \), conditionally to the information available in \((t-1)\), and \( v_t \) is the innovation process, with \( F_t \) as a covariance matrix.

- **Updating**:
  \[ \begin{align*}
  \tilde{\alpha}_t &= \tilde{\alpha}_{t+1|t} + P_{t+1|t} Z^T F_t^{-1} v_t \\
  P_t &= (I - P_{t+1|t} Z^T F_t^{-1} Z) P_{t+1|t}
  \end{align*} \]

The matrices \( T, c, R, Z, d, Q, \) and \( H \) are not time dependent in the simplest cases that we consider in this report. They are the system’s matrices associated with the state-space model.

---

\(^8\) Harvey (1989) and Roncally (1995) inspire this presentation.

\(^9\) \((\tilde{\alpha}_t - \alpha_t)\)^T is the transposed matrix of \((\tilde{\alpha}_t - \alpha_t)\).
1.3. The extended Kalman filter

When the model is non-linear, it is generally impossible to obtain an optimal estimator for the non-observable variables. An other method, the extended Kalman filter, can be used. However, it introduces an approximation in the estimation, because it leads to the linearization of the model.

In the non-linear case, the measurement and transition equations of the state-space form model are the following:

- Transition equation:
  \[ \alpha_{t|t-1} = T(\alpha_{t-1}) + R(\alpha_{t-1})\eta_{t} \]
  where \( \alpha_{0:t} \) is the state vector in \( t \), which is \((m \times 1)\), and where \( T(\alpha_{t-1}) \) and \( R(\alpha_{t-1}) \) are non-linear functions, depending on the values of the state variables in \((t-1)\).

- Measurement equation:
  \[ y_{t|t-1} = Z(\alpha_{t|t-1}) + \varepsilon_{t} \]
  where \( y_{t|t-1} \) represents multivariate temporal series \((N \times 1)\), and \( Z(\alpha_{t|t-1}) \) is a non-linear function of the non-observable variables.

As was the case in the simple Kalman filter, the two processes \( \varepsilon_{t} \) and \( \eta_{t} \) are supposed to be normally distributed, with zero mean, and with \( H \) and \( Q \) as the covariance matrices:

\[
E[\eta_{t}] = 0, \quad Var[\eta_{t}] = Q \quad E[\varepsilon_{t}] = 0, \quad Var[\varepsilon_{t}] = H.
\]

The system’s initial position is such as:

\[
[\alpha_{0}] = E \quad \text{and} \quad [\alpha_{0}] = Var \text{ associated with } \alpha_{t}.
\]

As a consequence, the following relationship defines the covariance matrix \( P_{t} \), associated with \( \alpha_{t} \):

\[
P_{t} = E[(\hat{\alpha}_{t} - \alpha_{t})(\hat{\alpha}_{t} - \alpha_{t})^{t}].
\]

- Linearisation:
  If the functions \( Z(\alpha_{t|t-1}) \) et \( T(\alpha_{t-1}) \) are smooth enough, it is possible to calculate their limited development around respectively \( \tilde{\alpha}_{t|t-1} \) and \( \tilde{\alpha}_{t-1} \), where \( \tilde{\alpha}_{t|t-1} \) is the expectation of \( \alpha_{t} \), conditionally to the information available in \((t-1)\), and \( \tilde{\alpha}_{t-1} \) is the value obtained for the state variable in \((t-1)\), at the end of the updating phase. The state-space linearized model is then:

\[
\begin{align*}
\hat{\alpha}_{t|t-1} & = \tilde{\alpha}_{t|t-1} + \hat{\varepsilon}_{t} \\
y_{t|t-1} & = \hat{Z}\tilde{\alpha}_{t|t-1} + \varepsilon_{t}
\end{align*}
\]

where:

\[
\begin{align*}
\hat{Z} & = \frac{\partial Z(\alpha_{t|t-1})}{\partial \alpha_{t|t-1}}_{\alpha_{t|t-1} = \tilde{\alpha}_{t|t-1}} \\
\hat{T} & = \frac{\partial T(\alpha_{t-1})}{\partial \alpha_{t-1}}_{\alpha_{t-1} = \tilde{\alpha}_{t-1}} \\
\hat{R} & = R(\tilde{\alpha}_{t-1}) \approx R(\alpha_{t-1})
\end{align*}
\]

In the extended version, the three steps of the iteration are the following:

- Prediction:
  \[
  \begin{align*}
  \tilde{\alpha}_{t|t-1} & = T(\tilde{\alpha}_{t-1}) \\
P_{t|t-1} & = \hat{P}_{t-1} \hat{T}^{t} \hat{R} + \hat{R} Q \hat{R}^{t}
  \end{align*}
  \]
  where \( \tilde{\alpha}_{t|t-1} \) et \( P_{t|t-1} \) are the estimators for \( \alpha_{0:t} \) and \( P_{0:t} \), conditionally to the information available in \((t-1)\).

- Innovation:
  \[
  \begin{align*}
  \tilde{y}_{t|t-1} & = Z(\tilde{\alpha}_{t|t-1}) \\
v_{t} & = y_{t} - \tilde{y}_{t} \\
F_{t} & = \hat{Z} P_{t|t-1} \hat{Z}^{t} + H
  \end{align*}
  \]
  where \( \tilde{y}_{t|t-1} \) is the estimation of the observation \( y_{t} \), conditionally to the information available in \((t-1)\), and \( v_{t} \) is the innovation process with \( F_{t} \) as a covariance matrix.

- Updating:
  \[
  \begin{align*}
  \tilde{E}_{t} & = \tilde{\alpha}_{t|t-1} + P_{t|t-1} \hat{Z}^{t} F_{t}^{-1} v_{t} \\
P_{t} & = (I - P_{t|t-1} \hat{Z}^{t} F_{t}^{-1} \hat{Z}^{t}) P_{t|t-1}
  \end{align*}
  \]

\(^{10}\) Harvey (1989) and Anderson and Moore (1979) inspire this presentation.
In the most simple case, the functions \( Z(\alpha_{t-1}) \), \( T(\alpha_{t-1}) \), and \( R(\alpha_{t-1}) \), just as the covariance matrices \( H \) and \( Q \), are not time dependent. \( Z(\alpha_{t-1}) \), \( T(\alpha_{t-1}) \) and \( R(\alpha_{t-1}) \) are the system’s functions. \( H \) and \( Q \) are the system’s matrices.

1.4. Kushner’s Filter

The initial process of the state variables is written:

\[
d\alpha(t) = T(\alpha(t))dt + \sigma(\alpha(t))dz(t)
\]

where \( z(t) \) is a brownian motion in \( \mathbb{R}^n \).

To be consistent with the previous notations the discretised version is the following:

\[
\alpha_t = \alpha_{t-1} + T(\alpha_{t-1})\eta_t
\]

with obvious correspondence between the terms.

It is supposed that the functions \( T \) and \( \sigma \) have the good regularity properties and that \( \alpha(0) \) has a density.

The observation are discrete and they are given by the following relation.

- **Measurement equation:**
  \[
y_j = Z(\alpha_j) + \epsilon_j
\]

  where the term \( \epsilon_j \) is gaussian with a null expectation and is independent of \( \alpha(0) \) and \( z(t) \), with covariance matrix \( Q \).

Kushner (1967) has shown that the conditional density of \( \alpha_k \), knowing the \( y_j \) for \( j=1,..k \), can be written in the following way:

\[
\rho_k(x) = c_k I_k(x) \tilde{\rho}_k(x)
\]

where:

- \( c_k \) is a normalisation constant,
- \( I_k(x) = e^{-\frac{1}{2}(x - Z(x))\eta(\gamma(x) - Z(x))} \)
- \( \tilde{\rho}_k(x) \) is the solution of the forward Kolmogorov equation associated with \( \alpha \) at time 1.

The initial condition at time 0 is the density \( \rho_{k-1}(x) \).

As generally, the Kolmogorov equation cannot be solved analytically. Therefore, we had to use complex numerical methods to evaluate \( \rho_k(x) \).

Considering that in general we only need to know the average of the distribution and possibly its variance, it seems natural to postulate the shape of the distribution - which is for example normal, as was the case in Kushner (1967). Then, we only need to compute the two first moments at each iteration.

- **Filtering step:** assuming the gaussian \( \tilde{\rho}_k(x) \) is known, it suffices to compute \( c_k \). Then we can calculate the 2 first moments of \( \rho_k(x) \).

- **Prediction step:** knowing these parameters, we can compute the two moments of \( \tilde{\rho}_{k+1}(x) \).

Clearly the filtering step requires three numerical integrations. Knowing the explicit form of the integrand, it is natural as is suggested in Kushner & Budhiraja (1997) to evaluate these integrals by gaussian quadratures at least for a small number of variables - says less than 4 - . This correspond to use the zeros \( x_i \) of the Hermite polynomials of degree n to approximate the integral. With this notation, the Gaussian quadrature can be written:

\[
\int_{-\infty}^{\infty} f(x) g(x) dx \sim \sum_{i=1}^{n} w_i f(x_i)
\]

where \( g(x) \) is the 0,1 is a gaussian density, and the weights \( w_i \) are chosen so that the formula is exact for polynomial of degree n. This imply that the formula will be exact up to the degree 2n-1.

The formula can be extended to higher dimensions if we consider the products of the previous sum. The practical limitation is that for an integral on \( \mathbb{R}^p \) we will have a sum of \( n^p \) terms. For high dimensions, it would be better to use Monte-Carlo methods.
The prediction step can be obtained using a linearization or better, a multistep gaussian quadrature.

### 1.5. Comparison of the two non linear filters

We evaluate the two filters, namely the extended filter and Kushner’s version, on a simple example. This example represents the kind of functions we could find in a commodity market: \( \alpha \) is an Ornstein-Uhlenbeck or mean reverting diffusion, like the convenience yield in the Schwartz’s model, and \( y \) is an exponential:

\[
y = \exp(a \alpha) + 0.4 \, N(0,1).
\]

The advantage of the example chosen is that we can easily simulate \( \alpha \) and as a results, it is easy to appreciate the performances of each version of the Kalman filter. It also allows us to make the nonlinearity vary with \( a \).

We first illustrate the effect of an increasing non linearity by making the parameter \( a \) vary from 0.2 to 1. The statistics for the expectation and the variance of the errors have been obtained on a series of 2000 points.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( E(X-X^*) )</th>
<th>( \text{Var}(X-X^*) )</th>
<th>( E(X-X^*) )</th>
<th>( \text{Var}(X-X^*) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.01</td>
<td>0.94</td>
<td>0.02</td>
<td>0.92</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.1</td>
<td>0.65</td>
<td>-0.06</td>
<td>0.60</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.23</td>
<td>0.47</td>
<td>-0.11</td>
<td>0.41</td>
</tr>
<tr>
<td>0.8</td>
<td>0.4</td>
<td>0.62</td>
<td>0.14</td>
<td>0.31</td>
</tr>
<tr>
<td>1</td>
<td>0.68</td>
<td>1.67</td>
<td>-0.12</td>
<td>0.26</td>
</tr>
</tbody>
</table>

The table 1 shows clearly that when \( a \) is small, the bias is small. The latter increases with \( a \). The estimation of the variance is quite high when the values of \( a \) are smaller or equal than 0.4, because the relative weight of the noise is large. These estimations of the variance also increase with the non linearity.

To illustrate the performances of the filters and to better understand what happens, we show on the figure 3 the 200 first points of the initial series and the estimations we obtained with the two versions of the Kalman filters.

![Figure 3. Comparison between the initial series (in blue) and its estimation using the Extended Kalman filter (in red)](image)
Figure 4. Comparison between the initial series (in blue) and its estimation using the Kushner’s version of the Kalman filter (in red)

Note that the scales are not the same between the two figures. To see even more clearly that, because we are not handling the non linearity completely, the measure forces the estimates to be skew, and high, we present a zoom of the two figures.

Figure 5. Zoom of the two preceding figures. a) the extended Kalman filter b) Kushner’s version.

1.6. The parameters estimation

Suppose now that the non observable variables and the errors and are normally distributed. Then the Kalman filter can be used to estimate the model’s parameters, which are supposed to be constants. On that purpose, we calculate, at each iteration and for a given vector of parameters, the logarithm of the likelihood function for the innovation $v_t$:

$$
\log I(t) = -\frac{n}{2} \times \ln(2\pi) - \frac{1}{2} \ln(dF_t) - \frac{1}{2} v_t^\top F_t^{-1} v_t
$$

where $F_t$ is the covariance matrix associated with the innovation $v_t$, and $dF_t$ its determinant\(^\text{11}\).

Relying on the hypothesis that the model’s measurement equation admits continuous partial derivatives of first and second order on the parameters, an other recursive procedure is employed to

\(^\text{11}\) The value of $\log I(t)$ is corrected when $dF_t$ is equal to zero.
estimate the parameters. An initial (M×1) vector of parameters is first used to compute all innovations of the study period and the logarithms of the likelihood function. Then the iterative procedure makes a search for the parameter’s vector \( x \) that maximizes the likelihood function \( f \) and minimizes the innovations. Once this optimal vector has been obtained, the Kalman filter is used, for the last time, to reconstitute the non observable variables and the measure \( \tilde{y} \) associated with these optimal parameters.

**Section 2. Applying the Kalman filter**

To explain how the Kalman filter can be used in finance, the filter is applied to a very famous term structure model of commodity prices, which was developed by Schwartz in 1997. The way to employ a Kalman filter in the case of term structure models is first explained. The Schwartz’s model is then presented, and we show how it can be transformed into a state-spaced model for a simple filter and for an extended filter. Once this has been made, we explain how the iteration process can be initiated, and how it can be stabilized.

### 2.1. The Kalman filter applied to term structure models

When the Kalman filter is applied to term structure models of commodity prices, the aim is the estimation of the measurement equation’s parameters, in order to obtain estimated futures prices for different maturities \( \tilde{F}(\tau) \), and to compare them with empirical futures prices \( F(\tau) \), as is shown in figure 6. The closest the firsts are with the seconds, the best is the model. So the way we use the Kalman filter is not perfectly straightforward, because the reconstitution of temporal series for non observable data is not the most important objective, and because the Kalman filter is always associated with an estimation method for the parameters. But there is still a need for the values of the non observable data to obtain the observable ones, which in that case, are the futures prices for different maturities. And the Kalman filter is a very fast mean to get them.

In the case of term structure models of commodity prices, the non observable data are, most of the time, the spot price \( S \) and the convenience yield \( C \). The later can be briefly defined as the comfort associated with the possession of physical stocks. There are usually no empirical data for these two variables, because there are most of the time no reliable time series for the spot price, and the convenience yield is not a traded asset.

The estimation of term structure models is not straightforward, because the analysis relies on two dimensions in time. The first dimension is the estimation period, between the 1st of September, 2000 and the 15th of August, 2002 for example. The second dimension is represented by the maturities of the futures contracts, for example the first, the third, the sixth and the ninth months of delivery.
The measure of the model’s performances must take into account these two dimensions. One way to appreciate these performances is to calculate the difference between $\tilde{F}$ and $F$ for different maturities, at one specific observation date, as is illustrated in figure 7. Here is appreciated the model’s ability, at one specific date, to represent the term structure of commodity prices. In the example represented on the figure 4, the innovation for the shorter maturity $\tau_i$ is smaller than the innovation for the longer maturity $\tau_n$ and the estimated futures prices, for all the maturity, present a positive bias: they are always superior than the empirical data.

**Figure 7. The estimation for different maturities at one specific date**

$$F(t, \tau_i) = F(\tau_i)$$

The second way to appreciate the model’s performances is to analyze the estimation’s error for one specific maturity $\tau_i$ and for the whole estimation period, as is show in figure 8. This time, the figure illustrates a negative bias in the estimation for the maturity $\tau_i$: for each date of the estimation sample, the estimated futures prices are always below the empirical data.

**Figure 8. The estimation for one specific maturity**

$$F(t_i, T) = F(\tau)$$

2.2. Schwartz’s model

The Schwartz’s model (1997) is one of the most famous term structure model of commodity prices. It presents three characteristics. First, its performances are good. Second, it has an analytical solution, which simplifies the application of the Kalman filter. Third, it allows for the use of a simple filter.
The Schwartz’s model supposes that two states variables, namely the spot price $S$ and the convenience yield $C$, can explain the behavior of the futures prices $F$. The dynamic of these state variables is the following:

\[
\begin{align*}
\frac{dS}{S} &= (\mu - C)dt + \sigma_S dz_S \\
\frac{dC}{C} &= [k(\alpha - C)]dt + \sigma_C dz_C \\
E[ dz_S \times dz_C ] &= \rho dt \\
\kappa, \sigma_S, \sigma_C > 0
\end{align*}
\]

with:

\[
\begin{align*}
\alpha & \quad \text{is the long run mean of the convenience yield} \\
\kappa & \quad \text{represents the convergence of the convenience yield towards } \alpha \\
\sigma_C & \quad \text{is the convenience yield’s volatility} \\
\rho & \quad \text{is the correlation between the two Brownian motions associated with } S \text{ and } C
\end{align*}
\]

The model’s solution expresses the relationship at $t$ between an observable futures price $F$ for a delivery in $T$, and the state variables. This solution is:

\[
F(S, C, t, T) = S(t) \times \exp \left[ -C(t) \frac{1 - e^{-\kappa T}}{\kappa} + B(\tau) \right]
\]

with:

\[
B(\tau) = \left[ \left( \tilde{\alpha} + \frac{\sigma_C^2}{2\kappa} - \frac{\sigma_S \sigma_C \rho}{\kappa} \right) \tau \right] + \left[ \frac{\sigma_C^2}{4} \frac{1 - e^{-2\kappa \tau}}{\kappa^3} \right] + \left[ \frac{\tilde{\alpha} \kappa + \sigma_S \sigma_C \rho - \frac{\sigma_C^2}{\kappa}}{\kappa} \left( \frac{1 - e^{-\kappa \tau}}{\kappa^2} \right) \right]
\]

where:

- $\tilde{\alpha} = \alpha - (\lambda / \kappa)$

2.3. Applying the simple filter to the Schwartz’s model

The simple filter is suited for linear models. To apply it, the solution of the Schwartz’s model can be easily expressed on a linear form, as follows:

\[
\ln(F(S, C, t, T)) = \ln(S(t)) - \frac{1}{\kappa} \ln(C(t)) + B(\tau)
\]

Considering the relationship $G = \ln(S)$, we also have:

\[
\begin{align*}
\frac{dG}{G} &= (\mu - C - \frac{1}{2} \sigma_S^2)dt + \sigma_S dz_S \\
\frac{dC}{C} &= [k(\alpha - C)]dt + \sigma_C dz_C
\end{align*}
\]

Then, to employ a Kalman filter, the model must be expressed on its state-space form. Its transition and its measurement equation characterize a state-space model.

The transition equation is the expression, in discrete time, of the state variables dynamic. Retaining the same notations as before, this equation is:

---

12 In that model, interest rates are supposed to be constant.
\[
\begin{bmatrix}
\tilde{G}_{i/t-1} \\
\tilde{C}_{i/t-1}
\end{bmatrix} = c + T \begin{bmatrix}
\tilde{G}_{i-1} \\
\tilde{C}_{i-1}
\end{bmatrix} + R \eta_t, \quad t = 1, \ldots, NT
\]

where:

- \(c = \left(\mu - \frac{1}{2} \sigma_s^2\right) \Delta t\) is a \((2 \times 1)\) vector, and \(\Delta t\) is the period separating 2 observation dates
- \(T = \begin{bmatrix} 1 & -\Delta t \\ 0 & 1 - \kappa \Delta t \end{bmatrix}\) is a \((2 \times 2)\) matrix,
- \(R\) is an identity matrix, \((2 \times 2)\),
- \(\eta_t\) are non correlated errors, with:
  \[E[\eta_t] = 0, \quad \text{and} \quad Var[\eta_t] = \begin{bmatrix} \sigma_s^2 \Delta t & \rho \sigma_s \sigma_C \Delta t \\ \rho \sigma_s \sigma_C \Delta t & \sigma_C^2 \Delta t \end{bmatrix}\]

The measurement equation is issued from the solution of the model:

\[\tilde{y}_{i/t-1} = d + Z \begin{bmatrix}
\tilde{G}_{i/t-1} \\
\tilde{C}_{i/t-1}
\end{bmatrix} + e_i, \quad t = 1, \ldots, NT\]

where:

- \(\tilde{y}_{i/t-1} = \left[\ln(\bar{F}(\tau_i))\right]\) is the \(i\)th line of the \(\tilde{y}_{i/t-1}\) vector for the estimated observable variables, with \(i = 1, \ldots, N\). \(N\) is the number of maturities which were retained for the estimation.
- \(d = \left[B(\tau_i)\right]\) is the \(i\)th line of the \(d\) vector, with \(i = 1, \ldots, N\)
- \(Z = \begin{bmatrix} 1 & -\frac{1 - \exp(-\kappa \tau_i)}{\kappa} \end{bmatrix}\) is the \(i\)th line of the \(Z\) matrix, which is \((N \times 2)\), with \(i = 1, \ldots, N\)
- \(e_i\) is a white noise’s vector, \((N \times 1)\), with no serial correlation: \[E[e_i] = 0 \quad \text{and} \quad H = Var[e_i]. \quad H \text{ is } (N \times N)\]

### 2.4. Applying the extended filter to the Schwartz’s model

From a practical point of view, passing from the simple to the extended filter implies that the system’s matrices \(Z, T\) and \(R\) are replaced with non linear functions, depending on the state variables. And to employ the extended Kalman filter, there is no need to express the Schwartz’s solution on a linear form.

The transition equation is directly issued from the dynamic of the state variables. In discrete time, keeping the same notations as before, this dynamic becomes:

\[
\begin{bmatrix}
\tilde{S}_{i/t-1} \\
\tilde{C}_{i/t-1}
\end{bmatrix} = T(\tilde{S}_{i-1}, \tilde{C}_{i-1}) + R(\tilde{S}_{i-1}, \tilde{C}_{i-1}) \eta_t,
\]

where:

- \(\begin{bmatrix}
\tilde{S}_{i/t-1} \\
\tilde{C}_{i/t-1}
\end{bmatrix}\) is the state vector, \((2 \times 1)\),
- \(T(\tilde{S}_{i-1}, \tilde{C}_{i-1})\) is a \((2 \times 1)\) vector:
  \[T(\tilde{S}_{i-1}, \tilde{C}_{i-1}) = \begin{bmatrix} \tilde{S}_{i-1} (1 + \mu \Delta t - \tilde{C}_{i-1} \Delta t) \\ \kappa \Delta t + \tilde{C}_{i-1} (1 - \kappa \Delta t) \end{bmatrix}\]
- \(R(\tilde{S}_{i-1}, \tilde{C}_{i-1})\) is a \((2 \times 2)\) matrix:
  \[R(\tilde{S}_{i-1}, \tilde{C}_{i-1}) = \begin{bmatrix} \tilde{S}_{i-1} & 0 \\ 0 & 1 \end{bmatrix}\]
- \(Q\) is a \((2 \times 2)\) matrix:
  \[Var(\eta_t) = \begin{bmatrix} \sigma_s^2 & \rho \sigma_s \sigma_C \\ \rho \sigma_s \sigma_C & \sigma_C^2 \end{bmatrix}\]

The measurement equation becomes:

\[\tilde{y}_{i/t-1} = Z(\tilde{S}_{i/t-1}, \tilde{C}_{i/t-1}) + e_i\]
where:
- \( \tilde{y}_{i, t-1} = \tilde{F}(\tau) \) is the \( i \)th line of the \( \tilde{y}_{i, t-1} \) vector for the estimated observable variables, with \( i = 1, \ldots, N \).
- \( Z(\tilde{s}_{i, t-1}, \tilde{c}_{i, t-1}) \) is a \((N \times 2)\) matrix. The \( i \)th line of this matrix is the following (\( i = 1, \ldots, N \)):

\[
Z(\tilde{s}_{i, t-1}, \tilde{c}_{i, t-1}) = \left[ \tilde{s}_{i, t-1} \times \exp(-H_i \tilde{c}_{i, t-1} + B(\tau)) \right]
\]

with:
\[
H_i = \frac{1 - e^{-xt_i}}{\kappa}
\]

\[
B(\tau) = \left[ r - \tilde{\alpha} + \frac{\kappa}{2\kappa^2} \tilde{\sigma}_s \tilde{\sigma}_c \rho \times \tau_i \right] + \left[ \frac{\kappa}{4} \times \frac{1 - e^{-2xt_i}}{\kappa^2} \right] + \left[ \left( \tilde{\alpha} + \sigma_s \tilde{\sigma}_c \rho - \frac{\kappa}{2} \right) \times \left( \frac{1 - e^{-xt_i}}{\kappa^2} \right) \right]
\]

\[
\tilde{\alpha} = \alpha - \lambda / \kappa
\]

- \( \epsilon_i \) is a white noise’s vector, \((N \times 1)\), with no serial correlation:

\[
E[\epsilon] = 0 \quad \text{and} \quad H = \text{Var}[\epsilon]. \quad H \text{ is } (N \times N)
\]

Lastly, the derivatives of the functions \( T \) and \( R \) conditionally to the state variables, respectively \( \hat{T} \) and \( \hat{R} \), are the following:

- \( \hat{T} \) is a \((2 \times 2)\) matrix:

\[
\hat{T}(\tilde{s}_{i, t-1}, \tilde{c}_{i, t-1}) = \begin{bmatrix} 1 + \mu \Delta t - \tilde{c}_{i, t-1} \Delta t & -\tilde{s}_{i, t-1} \Delta t \\ 0 & (1 - \kappa \Delta t) \end{bmatrix}
\]

- \( \hat{Z} \) is a \((2 \times N)\) matrix. The \( i \)th line of this matrix is the following, with \( i = 1, \ldots, N \):

\[
\hat{Z}(\tilde{s}_{i, t-1}, \tilde{c}_{i, t-1}) = \begin{bmatrix} e^{-H_i \tilde{c}_{i, t} + B(\tau)} \\ -H_i \times e^{-H_i \tilde{c}_{i, t} + B(\tau)} \end{bmatrix}
\]

where:
\[
H_i = \frac{1 - e^{-xt_i}}{\kappa}
\]

\[
B(\tau) = \left[ r - \tilde{\alpha} + \frac{\kappa}{2\kappa^2} \tilde{\sigma}_s \tilde{\sigma}_c \rho \times \tau_i \right] + \left[ \frac{\kappa}{4} \times \frac{1 - e^{-2xt_i}}{\kappa^2} \right] + \left[ \left( \tilde{\alpha} + \sigma_s \tilde{\sigma}_c \rho - \frac{\kappa}{2} \right) \times \left( \frac{1 - e^{-xt_i}}{\kappa^2} \right) \right]
\]

\[
\tilde{\alpha} = \alpha - \lambda / \kappa
\]

The extended Kalman filter is based on the linearisation of the function linking the observable variables to the non observables. Therefore, an approximation is made in this filter which is absent of the simple one.

### 2.5. Practical difficulties associated with the empirical study

To perform the empirical study, some difficulties must be overcome. First, there are choices to make when the iterative process is started. Second, if the model has been expressed in logarithm for the simple Kalman filter, some precautions must be taken when the performances are appreciated. Third the stability of the iteration process and the model’s performances are extremely sensitive to the covariance matrix \( H \).

#### 2.5.1. Starting the iterative process

To start the iterative process, there is a need for the initial values of the non observable variables and for their covariance matrix. Indeed, to proceed with the iteration’s prediction step at date 1, the values of the state variables and of the covariance matrix at date 0 must be known. Because the states variables are non observable, an approximation must be chosen.

In the case of the term structure models of commodity prices, the non observable state variables are, most of the time, the spot price and the convenience yield. The nearest futures price is generally retained as the spot price \( S \), and the convenience yield \( C \) is calculated with the solution of a simple term structure model, more precisely the Brennan and Schwartz’s model (1985). This solution requires the use of two observed futures prices, for delivery in \( T_1 \) and in \( T_2 \):
\[ c = r - \frac{\ln(F(S, t, T_1)) - \ln(F(S, t, T_2))}{T_1 - T_2} \]

where \( T_1 \) is the nearest delivery, and \( T_2 \) is just after.

The covariance matrix associated with the state variables must also be initialized. We choose a diagonal matrix, with the spot price’s variance and the convenience yield’s variance on the diagonal.

Once the approximation’s methods have been chosen, we had to decide which value to retain for the state variables and the covariance matrix. We choose the first value of the estimation period for the non observable variables, and we calculated the variances with the first 30 data of the estimation period.

To start the iterative process for the optimization, there is also a need for the parameters initial values. If the iteration process appears to be unstable, constraints can be added on the parameters.

### 2.5.2. Measuring the performances

When the solution of the model is expressed on its logarithmic form, some precautions must be taken when the model’s performances are measured. Indeed in that case, the innovations are calculated with the logarithm of the futures prices. Therefore there is a difficulty when the estimated and empirical data are rebuilt. The relationship linking the logarithm of the estimations \( \tilde{y}_{t/T-1} \) with the logarithm of the observation \( y_t \) is actually the following:

\[ y_t = \tilde{y}_{t/T-1} + \sigma R \]

where \( \sigma \) is the standard error of the innovations and \( R \) is a gaussian residue. To be perfectly correct, when the logarithm of the estimations is used to obtain the estimations themselves, the relationship between \( y_t \) and \( \tilde{y}_{t/T-1} \) becomes:

\[ e^{y_t} = e^{\tilde{y}_{t/T-1}} \times e^{\sigma R} \]

The expectation of the observation’s exponential is then:

\[ E[e^{y_t}] = E[e^{\tilde{y}_{t/T-1}}] \times e^{\frac{\sigma^2}{2}} \]

When the simple Kalman filter is applied to a model like the Schwartz’s model, when the estimated futures prices are compared with the empirical data, a corrective term should be added to the estimations exponential. The trouble is, this correction is delicate, because the innovations variance is modified as soon as the parameters change.

### 2.5.3. Stabilizing the iteration process

An other important choice must be made before initiating the Kalman’s iteration process. This choice concerns the estimation of the covariance’s matrix associated with the errors introduced in the measurement equation. This system’s matrix \( H \) is very important for the iteration’s stability, because it is added, during the innovation phase, to the innovations covariance’s matrix. In the simple Kalman filter, the relationship between the innovation’s matrix \( F_t \) and the system’s matrix \( H \) is actually the following:

\[ F_t = ZP_{t+1}Z' + H \]

where \( P_{t+1} \) is the covariance matrix associated with the non observable variables \( \tilde{\alpha}_t \), and \( Z \) is an other system’s matrix, included in the measurement equation.

During the next phase of the iteration process, the inverse of the innovation’s matrix is used for the updating of the non observable variables and their covariance matrix:

\[
\begin{align*}
\tilde{\alpha}_t &= \tilde{\alpha}_{t+1} + P_{t+1} \tilde{Z}_t F_t^{-1} v_t \\
P_t &= (I - P_{t+1} \tilde{Z}_t F_t^{-1} \tilde{Z}_t)' P_{t+1}^{-1}
\end{align*}
\]

Therefore, the updating of the non observable variable are strongly affected by the matrix \( H \). And if the terms of this matrix are too high, the iteration can become unstable.

Most of the time, the easiest way to estimate this matrix is to calculate the variances and the covariances of the estimation’s database. This method was retained to measure the model’s performances presented in the paragraph 3.3. But it is important to know how much the empirical results are affected by this choice. To show it, some simulations are presented in the paragraph 3.4.

---

\(^{13}\) \( e^{\tilde{y}_{t/T-1}} \) and \( e^{\sigma R} \) are not correlated.
Section 3. Comparison between the simple and the extended filters

The comparison between the performances of the Schwartz’s model measured with the two filters allows to appreciate the influence of the linearization on the results. In this section, the empirical data are first of all presented. Then the performance criteria are exposed. Finally, the results are delivered and commented.

3.1. The empirical data

The data used for the empirical study are daily crude oil prices for the settlement of the Nymex’s WTI futures contracts, between the 25th of September, 1995, to the 14th of January, 2002. They have been arranged such as the first futures price’s maturity \( \tau_1 \) is actually the one month’s maturity, and such as the second futures price’s corresponds to the two months maturity \( \tau_2 \), ... Keeping the first observation of each group of five, this daily data were transformed into weekly data. For the parameters estimation, and for the measure of the model’s performances, four series of futures prices\(^{14}\) were retained, corresponding to the one, the three, the six and the nine months maturities.

The interest rates are T-bill rates for a three months maturity. Because interest rates are supposed to be constant in the model, we used the mean of all the observations between 1995 and 2002.

3.2. The performances criteria

To measure the model’s performances, two criteria were retained: the mean pricing errors and the root mean squared errors.

The mean pricing errors (MPE) are defined in the following way:

\[
MPE = \frac{1}{N} \sum_{n=1}^{N} (\tilde{F}(n, \tau) - F(n, \tau))
\]

where \( N \) is the number of observations, \( \tilde{F}(n, \tau) \) is the estimated futures price for a maturity \( \tau \) at the date \( n \), and \( F(n, \tau) \) is the observed futures price. The mean pricing error is expressed in US dollar. It measures the estimation’s bias for one given maturity. If the estimation is good, the mean pricing error must be very close to zero.

Retaining the same notations, the root mean squared error (RMSE), expressed in US dollar, is defined in the following way, for one given maturity \( \tau \):

\[
RMSE = \left( \frac{1}{N} \sum_{n=1}^{N} (\tilde{F}(n, \tau) - F(n, \tau))^2 \right)^{1/2}
\]

The RMSE is an empirical variance. It measures the estimations stability. This second criteria is considered as the most representative, because prices errors can offset themselves and the mean pricing error can be low even if there are strong deviations.

3.3. The empirical results

The estimation period used to obtain the parameters are the following: from the 25th of September, 1995, to the 11th of May, 1998 and from the 18th of May, 1998, to the 15th of October, 2001. This period has different lengths (respectively 31, 5 and 53 months) because we wanted to measure the influence of the available information’s volume on the model’s performances. First, the optimal parameters obtained with the two filters are compared. Then the model’s ability to represent the prices curve and their dynamic is appreciated, on the learning database and on an expanded one. Finally, the sensitivity of the results to the errors covariance matrix is presented.

\(^{14}\) This means that \( N = 4 \) in the case we study.
3.3.1. Optimal parameters

The optimal parameters were estimated on two study periods with the simple and the extended filters. Their values are not the same\(^{15}\), as is illustrated by the tables 2 and 3.

### Table 2. Optimal parameters, 1995-1998

<table>
<thead>
<tr>
<th>Simple filter</th>
<th>Extended filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters</td>
<td>Gradients</td>
</tr>
<tr>
<td>Pull back force: ( \kappa )</td>
<td>1.969842</td>
</tr>
<tr>
<td>Trend: ( \mu )</td>
<td>0.142741</td>
</tr>
<tr>
<td>Spot price’s volatility: ( \sigma_S )</td>
<td>0.241347</td>
</tr>
<tr>
<td>Long run mean: ( \alpha )</td>
<td>0.089806</td>
</tr>
<tr>
<td>Convenience yield’s volatility: ( \sigma_C )</td>
<td>0.400676</td>
</tr>
<tr>
<td>Correlation coefficient: ( \rho )</td>
<td>0.967136</td>
</tr>
<tr>
<td>Risk premium: ( \lambda )</td>
<td>0.088951</td>
</tr>
</tbody>
</table>

During this first period, the optimal parameters obtained with the extended filter are most of the time higher than the ones associated with the simple filter. The principal differences concern the risk premium \( \lambda \) (110%), and the convenience yield’s long run mean \( \alpha \) (50%). This phenomenon does not reproduce itself during 1998-2001, as is shown in table 3: in that case, the differences between the two parameters series are lower, and the most important deviations are on the convenience yield’s volatility \( \sigma_C \) (26%) and the spot price’s volatility \( \sigma_S \) (23%).

### Table 3. Optimal parameters, 1998-2001

<table>
<thead>
<tr>
<th>Simple filter</th>
<th>Extended filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters</td>
<td>Gradients</td>
</tr>
<tr>
<td>Pull back force: ( \kappa )</td>
<td>1.59171</td>
</tr>
<tr>
<td>Trend: ( \mu )</td>
<td>0.379926</td>
</tr>
<tr>
<td>Spot price’s volatility: ( \sigma_S )</td>
<td>0.263525</td>
</tr>
<tr>
<td>Long run mean: ( \alpha )</td>
<td>0.252260</td>
</tr>
<tr>
<td>Convenience yield’s volatility: ( \sigma_C )</td>
<td>0.237071</td>
</tr>
<tr>
<td>Correlation coefficient: ( \rho )</td>
<td>0.93847</td>
</tr>
<tr>
<td>Risk premium: ( \lambda )</td>
<td>0.177159</td>
</tr>
</tbody>
</table>

The differences between the optimal parameters obtained with the two filters show, first that the linearization has a significant influence\(^{17}\), and second, that the parameters are not the same for different periods. In this study, the trend and the convenience yield’s long run mean are significantly higher for the second period.

### 3.3.2. The model’s performances

There are two ways to measure a model’s performances. The first uses the mean pricing error and the root mean squared errors to see how the model’s can duplicate the form of the term structure of futures prices. The second refers to graphics to show how the model reproduces the dynamic of the prices curve.

- **The ability to reproduce the form of the term structure of futures prices**

The first important conclusion of the study is that the model is able to reproduce the prices curve quite precisely, as in shown in the tables 4 and 5. For a nine month maturity, the mean pricing error is around USD 0.12 per barrel! And the RMSE is quite low, especially for the shorter period. The second conclusion is that if the RMSE is the relevant criteria, then the simple filter is always more precise than the extended one. This is true for the two periods, for all the maturities\(^{18}\). These measure also always decreases with maturity, which is consistent with Schwartz’s results on others periods. Nevertheless,

\(^{15}\) In the whole empirical study, optimisations have been made with a precision of \( 10^{-5} \) on the gradients.

\(^{16}\) For the two filters, and for the two periods, the parameters values retained to initiate the optimisation are the same. These values are the following: \( \kappa = 0.5 \); \( \mu = 0.1 \); \( \sigma_S = 0.3 \); \( \alpha = 0.1 \); \( \sigma_C = 0.4 \); \( \rho = 0.5 \); \( \lambda = 0.1 \).

\(^{17}\) Nevertheless, the parameters have the same order size that the one Schwartz obtained in 1997 on the crude oil market, on different periods.

\(^{18}\) The MPE and the RMSE presented in the table 3 can not directly be compared with the one Schwartz proposed in 1997, because this author has made the calculus with the logarithm of the futures prices.
Schwartz has worked with longer maturities, and shown that the root mean squared error increases again for deliveries after 15 months.

Table 4. The model’s performances with the simple and the extended filters, 1995-1998

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Simple filter</th>
<th>Extended filter</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MPE</td>
<td>RMSE</td>
</tr>
<tr>
<td>1 month</td>
<td>-0.063</td>
<td>1.2769</td>
</tr>
<tr>
<td>3 months</td>
<td>0.1064</td>
<td>1.1604</td>
</tr>
<tr>
<td>6 months</td>
<td>0.1453</td>
<td>0.0142</td>
</tr>
<tr>
<td>9 months</td>
<td>0.1419</td>
<td>0.8468</td>
</tr>
<tr>
<td>Average</td>
<td>0.0827</td>
<td>1.0796</td>
</tr>
</tbody>
</table>

Unit : USD/b.

The third conclusion is that the results obtained with the mean pricing errors are consistent with the previous one. The errors are always lower for the simple filter. Nevertheless, on the two periods, except for one maturity, the mean pricing errors have a general tendency to increase with the maturity. From 1995 to 1998, and for the two filters, they present a low positive bias, which turns into a negative one for the simple filter, during 1998-2001.

Table 5. The model’s performances with the simple and the extended filters, 1998-2001

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Simple filter</th>
<th>Extended filter</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MPE</td>
<td>RMSE</td>
</tr>
<tr>
<td>1 month</td>
<td>-0.060423</td>
<td>2.319730</td>
</tr>
<tr>
<td>3 months</td>
<td>-0.107783</td>
<td>1.989428</td>
</tr>
<tr>
<td>6 months</td>
<td>-0.054536</td>
<td>1.715223</td>
</tr>
<tr>
<td>9 months</td>
<td>-0.007316</td>
<td>1.567467</td>
</tr>
<tr>
<td>Average</td>
<td>-0.057514</td>
<td>1.897962</td>
</tr>
</tbody>
</table>

Unit : USD/b.

To be perfectly rigorous, the model’s performances associated with the simple Kalman filter should be corrected when, as is the case here, the logarithm of the estimations is used to obtain the estimations themselves (see 2.5.2.). The correction improves a little the performances, as is shown in table 6 : the root mean squared errors and the mean pricing errors diminish a bit for almost all the maturities.

Table 6. The comparison between the model’s performances associated with the simple filter, when there are or there are no corrections for the logarithm, 1998-2001

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Simple filter</th>
<th>Simple filter corrected</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MPE</td>
<td>RMSE</td>
</tr>
<tr>
<td>1 month</td>
<td>-0.060423</td>
<td>2.319730</td>
</tr>
<tr>
<td>3 months</td>
<td>-0.107783</td>
<td>1.989428</td>
</tr>
<tr>
<td>6 months</td>
<td>-0.054536</td>
<td>1.715223</td>
</tr>
<tr>
<td>9 months</td>
<td>-0.007316</td>
<td>1.567467</td>
</tr>
<tr>
<td>Average</td>
<td>-0.057514</td>
<td>1.897962</td>
</tr>
</tbody>
</table>

Unit : USD/b.

Finally, the innovations range diminishes with the futures contracts maturity, for the two periods. The figure 9 illustrates the innovations behavior for the one month’s maturity. It shows that they tend to return to zero, for the two periods and for the two filters. This is a good result, because this is what they are supposed to do in the Kalman filters. Nevertheless, as the figure illustrates it, even if the mean pricing errors are low for the two filters, the pricing errors, at certain specific dates, can be rather important. The maximum innovation in absolute value, for the extended filter, is USD 3,44 during 1995-1998, which represents 17% of the mean futures price for the one month maturity. For the simple filter, it is USD 3,21 or 15,86% of the mean futures price. For that period and for that maturity, the average of the innovations represents 0,4% of the mean futures price for a one month maturity for the extended filter, and 0,31% for the simple filter.

The maximum innovation increases a lot during the second period. In absolute value, during 1998-2001, it reaches USD 6 for the extended filter, which represents 25% of the mean futures price for a one month maturity. It is a bit lower for the simple filter : USD 5.

Therefore, as a conclusion, we can say that there is clearly an impact of the linearization introduced in the extended filter : it can be shown on the optimal parameters, on the performances, and on the innovations. Nevertheless, with an extended filter, the model’s ability to represent the prices curve is still good.
The ability to reproduce the dynamic of the term structure of futures prices

An other way to appreciate a model’s performances is to see if it is able to reproduce the price’s dynamic. This can be shown graphically.

On that point of view, the first important conclusion is that the model is able to reproduce the prices dynamic quite precisely, even if, like in 1998-2001, there are very large fluctuations in the futures prices. The figure 10 shows the results obtained for the one month’s maturity. During that period, the crude oil prices goes from USD 11 per barrel to USD 37 per barrel! Even if the Kalman filters are often suspected to be unstable, these results show that they can be used even with extremely volatile data. The graphic also shows that the two filters attenuate the range of prices fluctuations. This phenomenon can actually be observed for the two study periods, for every maturity.

The second important conclusion concerns the ability to reproduce the way prices curves evolve with time.

The figure 11 represents six term structures of crude oil prices, for different maturities, observed weekly on the Nymex between the 9th of August and the 14th of September, 1999. During this period, the prices curves are always in backwardation, and they are characterized by the presence of a little bump. Moreover, the intensity of the backwardation increases and the curve goes higher, as the futures prices for all the maturities rise.
The figure 12 shows how the model reproduced this evolution. It represents, for the same observations dates, the term structure of crude oil prices which were estimated with a simple Kalman filter. The model is able to replicate correctly not only the displacement towards the heights, but also the slope’s intensification. Finally, despite it is theoretically able to do it, the model doesn’t represent, in this example, the little bump in the curves that was empirically observed.

3.3.3. Expansion of the database

The parameters estimation shows, in 3.3.1, that they are not the same for different periods. Hence two questions arise. Firstly, is it necessary to often recalculate the parameters? Secondly, when does the calculus have to be done?

To bring a precise answer to these questions, a sensibility’s analysis of the estimated prices to the parameters should be undertaken. But measuring the model’s performances when the database is expanded and the parameters are kept the same as before can make a first step in the comprehension of what happens. This test has been made for two periods of three months, located in the prolongation of the two estimation’s periods: from the 18th of May to the 17th of August 1998 and from the 21st of October 2001 to the 14th of January, 2002.

One important conclusion issued from these tests is that the model’s performances decrease strongly when the database is expanded. The root mean squared errors and the mean pricing errors rise dramatically for the two periods. This phenomenon is particularly strong when the futures prices are volatile, during 2001-2002, and it will probably be even more pronounced as the database expansion’s length increase. Therefore there is a strong incentive to recalculate the optimal parameters each time the model is used. This is not especially an important drawback, at least when there is an analytical solution for the model, because then the estimation’s process is very fast.
Table 7. The model’s performances with an extrapolation on a three months period, in 1998

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Simple filter</th>
<th></th>
<th>Extended filter</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MPE</td>
<td>RMSE</td>
<td>MPE</td>
</tr>
<tr>
<td>1 month</td>
<td>2.0138</td>
<td>2.2012</td>
<td>1.7392</td>
</tr>
<tr>
<td>3 months</td>
<td>1.3296</td>
<td>1.3749</td>
<td>1.2448</td>
</tr>
<tr>
<td>6 months</td>
<td>0.6512</td>
<td>0.755</td>
<td>0.7563</td>
</tr>
<tr>
<td>9 months</td>
<td>0.2710</td>
<td>0.5442</td>
<td>0.4883</td>
</tr>
<tr>
<td>Average</td>
<td>1.0664</td>
<td>1.2188</td>
<td>1.0572</td>
</tr>
</tbody>
</table>

Unit : USD/b.

The differences in the performances we observe with the two filters are inverted when the optimal parameters of a given period are used to estimate futures prices on a period, which is situated after the learning period. The model is then most of the time more precise with the extended filter, and we observed this phenomenon for the two periods, as the tables 7 and 8 illustrate it.

Table 8. The model’s performances with an extrapolation on a 3 months period, in 2001-2002

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Simple filter</th>
<th></th>
<th>Extended filter</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MPE</td>
<td>RMSE</td>
<td>MPE</td>
</tr>
<tr>
<td>1 month</td>
<td>-0.710678</td>
<td>3.371702</td>
<td>-3.243584</td>
</tr>
<tr>
<td>3 months</td>
<td>-0.379108</td>
<td>2.972144</td>
<td>-2.920091</td>
</tr>
<tr>
<td>6 months</td>
<td>0.155104</td>
<td>2.500216</td>
<td>-2.247877</td>
</tr>
<tr>
<td>9 months</td>
<td>0.385290</td>
<td>2.164323</td>
<td>-1.767425</td>
</tr>
<tr>
<td>Average</td>
<td>-0.137348</td>
<td>2.750296</td>
<td>-2.544744</td>
</tr>
</tbody>
</table>

Unit : USD/b.

3.4. Simulations

The last results presented in this report are simulations. They show how the model’s performances are affected by the choice of the system’s matrix $H$. This matrix represents the errors in the measurement equation and the way it is estimated has a strong influence on the empirical results.

Most of the time, the terms of this matrix corresponds to the variances and the covariances of the estimation database, namely, in the case studied here, the variances and covariance between futures prices for different maturities. But one must know that the results obtained with the Kalman filter can be more precise if these terms are (artificially) lowered, as is shown in table 9. This table exposes the different results obtained during 1998-2001 with the extended Kalman filter. This period is especially interesting because the data fluctuate strongly The performances are achieved, first with the matrix based on the observations, then with artificially lowered matrices.

The simulations 1 to 4 correspond respectively to the model’s performances obtained by multiplying the system’s matrix $H$ by (1/2), (1/16), (1/160), and (1/1600). As the matrix is lowered, the model’s performances improve strongly: from the initial performances to the fourth simulation, the root mean squared error is almost divided by two. The comparison between the third and the fourth simulation also illustrates the fact that there is a limit to the performances amelioration.

Table 9. Simulations with different system’s matrix $H$

<table>
<thead>
<tr>
<th>Observations</th>
<th>1 month</th>
<th>3 months</th>
<th>6 months</th>
<th>9 months</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>MPE</td>
<td>0.0979</td>
<td>0.0573</td>
<td>0.1096</td>
<td>0.1412</td>
<td>0.1015</td>
</tr>
<tr>
<td>RMSE</td>
<td>2.2945</td>
<td>2.1207</td>
<td>1.8777</td>
<td>1.6952</td>
<td>1.9970</td>
</tr>
</tbody>
</table>

Simulation 1

| MPE          | 0.0013  | 0.0935   | 0.1501   | 1.6506   | 0.4739  |
| RMSE         | 1.8356  | 1.5405   | 1.2478   | 2.6602   | 1.8210  |

Simulation 2

| MPE          | 0.0073  | 0.0152   | 0.0612   | 0.0137   | 0.0244  |
| RMSE         | 1.4759  | 1.1686   | 0.9386   | 0.8317   | 1.1037  |

Simulation 3

| MPE          | 0.0035  | -0.0003  | 0.0383   | 0.0005   | 0.0105  |
| RMSE         | 1.3812  | 1.0950   | 0.8647   | 0.7499   | 1.0227  |

Simulation 4

| MPE          | 0.0131  | 0.0067   | 0.0415   | 0.0075   | 0.0172  |
| RMSE         | 1.3602  | 1.0919   | 0.8697   | 0.7591   | 1.0202  |
The figure 13 portrays the main results of these simulations.

**Figure 13. One month’s futures prices observed/estimated, 1998 - 2001**

Section 4. Conclusion of the third part

The Kalman filters are powerful tools, which can be employed for model’s estimation in many areas in finance. They are especially well suited for finance because they are fast even if they have to deal with a large amount of information and because they allow for unobservable variables. Moreover, they can be used for linear as well as non linear models, even if there is no analytical solution for the models.

The main conclusions of this third part are the following. First, the extended Kalman filter introduces an approximation, which is due to the model’s linearization. This approximation has clearly an influence on the model’s performances: the extended filter leads generally to less precise estimations than the simple one. Nevertheless, the difference between the two filters is quite low and the extended filter is still acceptable. The second conclusion is that the estimations results are sensible to the system’s matrix containing the errors of the measurement equation and that this matrix can be used to obtain more precise results on the estimation base. The third important conclusion is that at least for the term structure models of commodity prices, the parameters are not constant in time and there is a need to recalculate them very often. This can become a problem if the model has no analytical solution, because of the computing time. Lastly, the approximation made in the extended Kalman filter is not a real problem until the model becomes really non linear. In that case, some other methods may be used, like the one Küchner (1968) proposed. The study of this method, also well suited for non linear models, constitute the natural prolongation of this work.
Conclusion of the report

At the end of this report, we would like to explain what we think about the main interests of this study, and to expose what we intend to do in the future.

The whole study presented here is situated in the prolongation of earlier researches, but the first part of the report, which is concentrated on the analysis of futures prices curves, is entirely new. Its main advantage, to our opinion, is that it relies on a particularly interesting database. This database allowed us to study the behavior of futures prices for very long maturities, as far as seven years. This part of the work showed mainly that the movements of the prices curves are quite simple to describe, and that the curve is divided into two parts: the shorter and the longer one. Different underlying factors can be evoked to explain these two parts of the curve.

The second part is directly linked to prior articles (Lautier and Galli, 2000, Lautier 2002), which are added to the report. It is centered on the asymmetrical model and it aims to reduce the calculus time associated with a futures price. The final version of this second part was not the same in the preliminary report. Meanwhile, we find another way to gain in computing time, and the new solution we propose now is faster.

In the third part of the report, we exploited some earlier researches on the Kalman filters, especially on the extended Kalman filter. These methods are particularly useful in the commodity markets, because they allow for non observable variables, which are common in this markets. We were the first, in 2000, to apply this filter in the finance field. Relying on this earlier work, we extended our knowledge on the extended filter and its performance. We also developed, in this report, another new method, the Kushner filter, and we compared it, on a simple example, with the extended Kalman filter.

In the future, we intend to employ this report in the following way. First of all, we would like to use this work for international publications. We could probably write two articles with the report: the first one based on the first part, and the second one based on the third part.

We also think to the continuation of this work.

We would like to use the second part of the report to test the performances of the asymmetrical model on the crude oil market, and to compare them with the performances of the Schwartz model. Some other commodities markets, like the cooper market or the electricity markets, could also be explored with these two models. Last but not least, we envisage the application of the model to the study of hedging strategies. If the asymmetrical model is more able to reproduce the futures prices curve than the Schwartz model, then our model should lead to more efficient hedging strategies.

The third part of the report could also be improved. We aim to apply the Kushner filter to a term structure model of commodity prices and to determine the way we could estimate the parameters with this filter.
References


HULL J.C., 2000, Options, futures, and other derivatives, Prentice Hall, p 357-360.


